

# Long Run Risk and the Persistence of Consumption Shocks

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## ABSTRACT

In this paper we supply a new long-run risk valuation model in which the pricing of long-run risk in consumption growth is made consistent with a more realistic description of the persistence and predictability properties of all the relevant time series. The main innovation is the use of a decomposition of time series which classifies innovation shocks on the basis of their half-life. Correspondingly the relation between price variations and the persistent fluctuations in both consumption and cash flows growth is disaggregated across different levels of persistence and the complete term structure of risk-return trade-off is computed. Remarkably the empirical tests performed within our model remove most of the counterfactual implications generated by the original R.Bansal and A.Yaron (2004) model. In particular we find that consumption growth does contain cyclical components that are predictable, while it remains unpredictable at the aggregate level. These predictable components, moreover, are highly correlated with well known structural drivers of consumption variability, such as long-run productivity growth and demographic effects. The estimation of the term structure of risk premia produces evidence that these long-run drivers of consumption are priced by the market and contribute significantly to explain the equity premium. Finally, we propose a new test on the elasticity of intertemporal substitution which properly accounts for the persistence heterogeneity in both the risk free rate and the consumption growth and produces an estimate significant and larger than one.

*JEL classification:* G12, E21, E32, E44

*Keywords:* Long-run risks models; Persistence heterogeneity; Temporal aggregation; Consumption predictability.

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## I. Introduction

This paper contributes to the ongoing research on long-run risk in asset pricing. The contribution consists in a consumption-based asset pricing model in which long-run risk in consumption growth is priced and contributes to the equity premium as in Bansal and Yaron (2004) and yet the model is free of the empirically unsupported implication that consumption growth is predicted by the price-dividend ratio (see Constantinides and Ghosh (2008) and Beeler and Campbell (2009)).

Our extension is motivated by the empirical observation that consumption growth is generated both by *predictable* low frequency variations and by *non-predictable* highly volatile, high frequency idiosyncratic variations. This empirical evidence that motivates our extension is obtained by disaggregating consumption growth into cyclical components, classified by their level of persistence (or characteristic half-life). While consumption growth remains unpredictable at the aggregate level, it does contain cyclical components that are predictable. These predictable components, moreover, are highly correlated with well known structural drivers of consumption variability. On the longest side, for instance, demographic shocks are highly correlated with the component describing consumption growth variations that occurs on time scales which range between 16 and 32 years. On the intermediate side, long-run productivity growth explains cyclical variations with time scale between 8 and 16 years. On the shortest side, finally, an high frequency predictable component with a yearly half-life is found and can be identified with the well documented fourth quarter effect (see Moller and Rangvid (2010)).

As the above mentioned empirical evidence suggests, it is therefore important to develop an asset pricing model where consumption responds to shocks of heterogeneous durations. Inspired by this observation and to study the effect of such diversity, we introduce a parsimonious equilibrium model where a representative agent with Epstein-Zin preferences faces an exogenous consumption and dividend streams driven by many factors, each one operating over different time horizons. Given our preference choice, these same factors enter the stochastic discount factor of the agent thus affecting asset returns. Our model differs from the standard long-run risk along three main dimensions. First, whereas in the standard long-run risk model stock prices respond strongly to variation in future aggregate consumption and dividend growth, in our model this relation between price variations and the fluctuations in both consumption and cash flows growth is disaggregated across different levels of persistence in order to make the model consistent with the (lack of) empirical evidence of aggregate consumption predictability. Second, while predictability is induced exogenously by latent factors, our model identifies “observable” well-known drivers of specific components of consumption growth which are classified by their level of persistence. Third, since

in our model the stochastic discount factor dynamics are driven by shocks with highly heterogeneous durations, we are able to characterize the dependence of the price of risk on the investment horizon and to reconstruct the entire term-structure of the risk-return trade-off.

The model builds on a decomposition of time series into the sum of components with different persistence levels.<sup>1</sup> This decomposition allows us to disaggregate consumption and dividend growth into different components each one driven by its own state variable. Consistent with the idea that each state variable operates at a specific frequency we model their dynamics with a multiscale autoregressive process, i.e. each autoregressive operates over a time interval of increasing length and its autoregressive coefficient uniquely identifies the persistence of the shocks. With this specification for the dynamics of the state variables our model is extremely parsimonious and tractable and yields closed-form expressions for equilibrium prices and return dynamics.

This approach generates very important implications for consumption predictability. In fact the presence in the same time series of highly persistent components with small volatility together with highly volatile components with low persistence can hide the predictable relation generated by the most persistent ones. From this point of view we interpret the findings of Beeler and Campbell (2009) as indicating the absence of predictability only for the low persistence components of consumption growth which are likely to provide the largest contribution to aggregate consumption volatility. On the other hand once we properly disaggregate the time series of interests across different levels of persistence, we do find that price components reflect future prospects of consumption components, as predicted by our model.

Our model classifies the shocks impinging the economy along two competing dimensions: their size as measured by their instantaneous volatility and their persistence as measured by their half life. Controlling simultaneously for these two dimensions we are able to obtain an interesting decomposition for the equity premium across different time horizons. In particular the term structure of equity premium implied by our model allows us to conclude that high frequency components which are responsible for most of the consumption growth variance produce a negligible contribution to equity premia; on the contrary low frequency, thin (in variance) components account for most of the long-run risk contribution to equity risk premia. Intuitively in the short-run the effect on prices of slow moving structural changes like those induced by technological innovation or by demographic trends, is completely hidden by the myopic and volatile reaction of markets to the incoming flow of information. However as the

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<sup>1</sup>Whereas in this paper we concentrate on the valuation implications of such an interpretation, a formal discussion of the econometric methodological issues required to extract these components can be found in the companion paper OTT2010b.

valuation horizon increases, the effect of transitory shocks is averaged out while persistent structural trends emerge as the driving forces of long-run expectations and play a pivotal role in the rational valuation of assets. This suggests that our classification of shocks is potentially critical for asset valuation.

Finally, this paper contributes to the debate on the estimates of the intertemporal elasticity of substitution (IES) of the representative agent. By separating the consumption growth into cyclical components we obtain an estimate of the IES which is strictly greater than one and the Bansal and Yaron (2004) hypothesis that the substitution effect dominates the income one, which means that the elasticity of intertemporal substitution must be greater than one, is empirically supported. If instead we do not account for persistence heterogeneity our estimated equation collapses to the one in Beeler and Campbell (2009) where the data provide strong statistical evidence against an IES greater than one.

The above results provide strong empirical support to the following conclusion: the decomposition of the time series into cyclical components classified by their level of persistence identifies predictable patterns present in aggregate consumption and dividends growth data which cannot be detected using traditional models. Importantly these predictable variations in consumption and cash flows are priced by the market and contribute significantly to the explanation of the equity premium.

The remainder of the paper is organized as follows. The next subsection concludes the introduction with a review of the literature. Section **II.A.** revisits Bansal and Yaron (2004) to show briefly that long-run risk being priced generates the counterfactual implication that the consumption growth and the price-dividend ratio have the same level of persistence. This motivates the needs to find a viable method to separate a time series in components characterized by their levels of persistence. Section **II.B.1.** introduces quickly such a method and then Section **II.B.2.** uses it to filter the time series of interests in our long-run risk asset pricing model which accounts for persistence heterogeneity. Section **III.** explores the main empirical findings obtained applying the new persistence based decomposition. Section 4 concludes.

#### *A. Related literature*

Our research contributes to the fast growing stream of literature which looks at the long-run regime to explain many of the inconsistencies which affect predictions of dynamic asset pricing models.

In their seminal contribution to long run risk valuation Bansal and Yaron (2004) explain stock price variations as a response to small persistent fluctuations in the mean and volatility

of aggregate consumption growth by an agent with elasticity of substitution greater than one and recursive preferences a la Epstein Zin (Kreps and Porteus 1978, Epstein and Zin 1989, Epstein and Zin 1991). The long-run risk theoretical framework has motivated many empirical tests on the presence of long run consumption risk in the data.<sup>2</sup> For example Parker and Julliard (2005) find that the consumption CAPM performs better at predicting the cross sectional differences if one uses long-run consumption growth rates instead of short run ones. Along the same line Bansal, Dittmar and Lundblad (2005) show that long-run risks in cash flows are an important risk source in accounting for asset returns and Bansal, Dittmar and Kiku (2007) show that economic restrictions of cointegration between asset cash flows and aggregate consumption have important implications for cross-sectional variation in equity returns, particularly for long horizons.

Nevertheless some recent papers, namely Constantinides and Ghosh (2008) and Beeler and Campbell (2009), evaluate the long-run risks model and find reversal of earlier conclusions. The main contribution of our paper is to reconcile these evidence within a theoretical framework which still allows for long-run risk in consumption growth.

Other recent and interesting implementations of long-run risk model try to understand the source of persistent predictable component in consumption growth. With this respect Garleanu, Panageas and Yu (2009) focus on the impact of major technological innovations and real options on consumption and the cross-section of asset prices. These innovations are assumed to occur at a very low frequency (greater than ten years), and are shown to carry over into a small, highly persistent component of aggregate consumption. Analogously Kaltenbrunner and Lochstoer (2010) and Croce (2010) show that consumption and savings decisions of agents in a production economy lead to low-frequency movements in consumption growth that are linked to the conditional mean of productivity growth. Similar to these authors we find that shifts in the long-run rate of productivity growth are one of the key factors driving the slow-moving consumption components.

With regard to the estimation of the intertemporal elasticity of substitution, the empirical literature has produced contradictory evidence on this point. On one hand Hall (1988) and Campbell and Mankiw (1989) estimated an extremely small value of IES on US data and Campbell (2003) summarizes these results and finds similar patterns in international data. On the other hand Attanasio and Weber (1993) and Beaudry and van Wincoop (1996) have found values of the IES higher than one using disaggregated cohort-level and state-level consumption data. Vissing-Jorgensen (2002), moreover, pointing out that many consumers

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<sup>2</sup>Recent literature focuses on the asset pricing implications in equity, bond and currency markets refer, for instance, to Bansal and Shaliastovich (2010) and Koijen, Lustig, Nieuwerburgh and Verdelhan (2010b). We do not pursue this line of research further and we just focus on the equity markets.

do not participate actively in asset markets, finds that, in household data, the IES is greater than one for asset market participants. Similar to the latter studies we present empirical evidence on aggregation problems with the relationship between consumption growth and the real interest rate and we show that using disaggregated consumption data is key to find a value for the IES greater than one. However we add to this literature since the main driver of our results is the intrinsic persistence heterogeneity in consumption and not the differences in preferences and/or opportunity sets for different cohort or the different levels of stock market participation.

Finally our work, which builds on a decomposition of time series in a sequence of shocks classified by their level of persistence, is close to Calvet and Fisher (2007) who investigate the role of heterogeneity in persistence of volatility in a partial equilibrium set-up by means of non linear regime switching multifractal models. Our technique can also be related to the multiplicative permanent-transitory decomposition proposed in Hansen and Scheinkman (2009) and used in Hansen, Heaton and Li (2008) and Alvarez and Jermann (2005). For a formal analysis of the link between these two spectral approaches we refer the interesting reader to OTT2010b.

## II. A Long Run Risk Model with Heterogeneous Persistence

In this section we first revisit Bansal and Yaron (2004) (BY04 henceforth) to show briefly why long-run risk being priced generates the counterfactual implication that the consumption growth and the price-dividend ratio have the same level of persistence. Motivated by this fact we then introduce our long-run valuation model which accounts for persistence heterogeneity. We focus our attention on the case in which second moments are constant although the further extension to stochastic volatility would entail no formal impediments. By assuming constant volatilities of log consumption growth and log dividend growth we are able to better concentrate our attention on the primary research question under debate, that is whether fluctuations in the conditional mean of consumption and dividend growth are indeed priced.

### A. Long-run risk versus consumption and price-dividend persistence

We consider a simplified version of the BY04 in which aggregate consumption is equal to the aggregate dividend.<sup>3</sup>

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<sup>3</sup>BY04 introduce a leverage effect between dividends and consumption in order to allow dividend growth to be more volatile than consumption growth and to allow for an imperfect correlation between consumption growth and dividend growth, as it is in the data. Our point holds regardless of this effect being present.

The data generating process for consumption is then as follows:

$$\begin{aligned} g_{t+1} &= \mu + x_t + \sigma\eta_{t+1} \\ x_{t+1} &= \rho x_t + \phi_e \sigma e_{t+1} \\ \eta_{t+1}, e_{t+1} &\sim i.i.d.N(0, 1) \end{aligned} \tag{1}$$

where  $g_{t+1}$  is the log rate of consumption growth and  $x_t$  is the state variable of the model. Relying on log-linear approximations for the log return on the market portfolio BY04 show that, in equilibrium, the log price-dividend ratio  $z_t^m$  is an affine function of the state variable, i.e.

$$z_t^m = A_{0,m} + A_{1,m}x_t \tag{2}$$

and the equity premium is given by

$$E[r_{m,t+1} - r_{f,t}] + 0.5var(r_{m,t}) = \gamma\sigma^2 + (1 - \theta)\kappa_{1,m}^2 A_{1,m}^2 \phi_e^2 \sigma^2 \tag{3}$$

Observe now that, in equation (3), the long-run risk contribution to the expected excess returns is proportional to  $A_{1,m}$ . Thus, as long as the representative agent has non-trivial Epstein-Zin preferences, i.e.  $\theta \neq 1$ , long-run risk is priced in the BY04 model if and only if  $A_{1,m}$  is different from zero. Considering equations (1) and (2) together, therefore, long-run risk is priced if and only if the consumption growth and the price-dividend ratio have the same decay rate in the autocorrelation function<sup>4</sup>, i.e. the same persistence.

This implication of the model however is empirically rejected. On the one hand, in fact, the price-dividend is (close to) a unit root process, as it is well documented in the literature, see e.g. Torous, Valkanov and Yan (2004), Campbell and Yogo (2006) and Lettau and Nieuwerburgh (2008). On the other hand consumption growth resembles closely a white noise process and therefore does not share the high persistence of the price-dividend ratio. This point is well synthesized in Figure 1 that displays the very different statistical behavior of the demeaned price-dividend and consumption growth series. The figure shows that over the sample 1947Q2-2009Q4 the price-dividend ratio has crossed its mean value much less often

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<sup>4</sup>This follows immediately from the fact that, given the equations in (1), the expression for autocovariance of the consumption growth at lag  $k$  is given by

$$cov(g_t, g_{t+k}) = \frac{\rho^k \phi_e^2 \sigma^2}{1 - \rho^2}$$

and as long as  $A_{1,m} \neq 0$  then the covariance for the price-dividend ratio will decay as  $\rho^k$  as well.

than the consumption growth. In fact Campbell and Shiller (2001) report that the price-dividend ratio has crossed its mean value only 29 times since 1872. The intervals between crossings for the price-dividend ratio range from one year to twenty years, the twenty-year interval being the one between 1950 and 1970. The persistence of the consumption growth is only moderate however, the half life of consumption growth shocks being 1 year.<sup>5</sup>

[Insert Figure 1 about here.]

Not surprisingly the empirical literature has tested and rejected other implications of the BY04 model. Constantinides and Ghosh (2008), for instance, examine the ability of the model to explain the returns of the market portfolio and find that the average pricing error is substantial. This fact can be easily understood if one recalls that in order to price the average equity returns the stochastic discount factor must have a large permanent component (see Alvarez and Jermann, 2005 and Koijen, Lustig and Van Nieuwerburgh, 2010a). In BY04 however the persistence of the stochastic discount factor, which depends on the price-dividend ratio<sup>6</sup>, is tied to the low persistence of the consumption growth. It is then not surprising that using this constrained pricing kernel they find that the model is rejected at the annual frequency over the 1930-2006 sample. Beeler and Campbell (2009), moreover, test the model by evaluating the ability of the log price-dividend ratio  $z_t^m$ , proxying for the latent state variable  $x_t$ , to predict consumption growth. As noted in Figure 1 however, consumption growth is, at quarterly horizon, close to a white noise and therefore it would be difficult to find evidence of a predictable persistent component at this frequency of observations. Not surprisingly the simple OLS regressions of consumption growth on the log price-dividend ratio run by Beeler and Campbell (2009) display relatively little, if any, predictability of consumption growth in the data.

A possible way to reconcile the findings of Beeler and Campbell (2009) with the long-run risk framework is to think of consumption growth as the sum of components with different levels of persistence, where the highly persistent ones contribute for a very small fraction to the total volatility of aggregate consumption growth and yet are predictable by the highly per-

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<sup>5</sup>Similarly Paseka and Theocharides (2010) find that, by relaxing the equilibrium restriction (2) that requires the price-dividend to be affine in the long-run risk variable, the persistence of the latent mean consumption growth corresponds to an half life of about 1.3 years for the 1934 – 2005 period.

<sup>6</sup>In our simplified version of BY04 the log pricing kernel can be expressed in terms of observables, namely the aggregate log price-dividend ratio, and its lags, and consumption growth. In the general version of the BY04 model, the aggregate log price-dividend ratio and log interest rate are affine functions of the long-run risk variable and the conditional variance of its innovation. In this more general case Constantinides and Ghosh (2008) show that it is possible to express the log pricing kernel as an affine function of the aggregate log price-dividend ratio, log interest rate, and their lags, in addition to consumption growth. As we said in this paper we do not consider stochastic volatility since our interest is just in the the long-run risk channel and not in macroeconomic uncertainty.



sistent components of the financial ratios. If this was the case the contemporaneous presence in the same time series of highly persistent components with small volatility together with highly volatile components with low persistence would generate a severe errors-in-variable problem in the Beeler and Campbell (2009) regression, a problem hiding the predictability relation which could eventually hold for specific components.<sup>7</sup> In order to see if this alternative way of interpreting the time series could help reconciling the above results, we first need to develop a tool to decompose the time series of interest into components with different levels of persistence. We briefly introduce this tool in the next section and then we use it in our asset pricing model to account for persistence heterogeneity.

## *B. Long-run valuation and Persistence Heterogeneity in Consumption*

### *B.1. A Persistence-based Decomposition of Time Series*

The above discussion highlights the necessity to classify the components of a time series on the basis of their level of persistence. To give the basic intuition behind our decomposition we can think of applying a  $h$ -period moving average to our time series. The averaging action smooths the original time series by removing the high frequency components. The output of this moving-average filter is a new time series which conveys information only about those cyclical components with periodicity greater than  $h$  periods. Of course, by increasing the window length  $h$  of the moving-average filter we would be able to extract components that decay more and more slowly. This procedure, however, imposes only a lower bound on the persistence of the filtered components whereas our aim is to obtain a component with a well-defined level of persistence. Ideally, in fact, we would like to isolate those fluctuations of the original time series that lie within a specific band of frequencies. To do so one can consider the outputs of two moving-average filters with windows  $h > h'$ . Since the moving average filters yield two time series, one characterized by fluctuations longer than  $h'$  periods and one by fluctuations longer than  $h$  periods, the difference between these two time series should in principle identify the component reflecting the fluctuations of the original time series with periodicity between  $h'$  and  $h$  periods.

To lay down some basic notation useful in what follows, given a time series  $\mathbf{x} = \{x_t\}_{t \in \mathbb{Z}}$

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<sup>7</sup>We come back to this point in Section **III.B.** and **III.C.**

consider its sample mean over a window of past observations with size  $2^j$ :<sup>8</sup>

$$\pi_t^{(j)} = \frac{1}{2^j} \sum_{p=0}^{2^j-1} x_{t-p} \quad (4)$$

where  $j \geq 1$  and  $\pi_t^{(0)} \equiv x_t$ . Following the intuition given above, the component  $x_{j,t}$  that identifies the fluctuations of the time series  $\mathbf{x}$  with periodicity between  $2^{j-1}$  and  $2^j$  periods<sup>9</sup> is then filtered out as follows:

$$x_{j,t} = \pi_t^{(j)} - \pi_{t-2^{j-1}}^{(j-1)} \quad (5)$$

In the sequel of the paper we refer to  $j$  as the level of persistence of the component  $x_{j,t}$ . Since in the empirical work one always deals with time series of finite length  $T$ , we also define the maximum (observable) level of persistence  $J$  as the the greatest integer such that  $2^J \leq T$ . Clearly given the data sample it is impossible to draw any inference about the persistence of shocks that last longer than  $2^J$  periods.<sup>10</sup>

Simple algebra shows that, for any given  $J$ , the generic element  $x_t$ , the components  $\{x_{j,t}\}_{j=1}^J$  and the permanent part  $\pi_t^{(J)}$  are related via the following identity:<sup>11</sup>

$$x_t = \sum_{j=1}^J x_{j,t} + \pi_t^{(J)} \quad (6)$$

This relation fits well with the intuition that current financial and economic quantities are the result of the overlay of past fluctuations with different periodicity, where these fluctuations can go from the extreme of an incoming flow of information at high frequency (low  $j$ ) to the one of slowly moving structural changes (high  $j$ ) like those induced by technological innovation or demographic trends. In Section **III.D.** we foster this idea by showing that the predictable consumption components are indeed highly correlated with well known structural drivers of consumption variability.

The components  $\{x_{j,t}\}_{j=1}^J$  can also be used to obtain an alternative decomposition of the

<sup>8</sup>Note that the window of values over which the average is made increases exponentially with base two. The moving average filters with such a property are optimal in the sense that they satisfy the principles of multiresolution analysis (see e.g. Mallat (1989a), Mallat (1989b), Daubechies (1990) and Daubechies (1992)). See also OTT2010b.

<sup>9</sup>For an interpretation of the cycle durations corresponding to the persistence level  $j$  in the case of quarterly time series see Table **IV.**

<sup>10</sup>In Appendix **AB.** we relate the persistence properties of the series  $x_{j,t}$  and  $\pi_t^{(J)}$  to their Fourier spectra.

<sup>11</sup>The algebra behind relations (6) and (7) is carried out in Appendix **AA.**

original time series, i.e.

$$x_{t+1} = \sum_{j=1}^J x_{j,t+2^j} + \pi_{t+2^J}^{(J)} \quad (7)$$

Intuitively this decomposition, which in the sequel will be referred as the forward decomposition, can be interpreted as a way to reconstruct the realization of  $x_{t+1}$  from the effect that this realization will have at different horizons and it will turn out to be a fundamental tool in working out our model.

A last comment is in order. In general the component  $\pi_t^{(J)}$  is meant to capture the finite sample permanent component of the time series, in the sense that it conveys information about all those cyclical components with periodicity greater than  $2^J$  periods including, potentially, the truly permanent component.<sup>12</sup> However in the rest of the paper we will focus on stationary time series for which it turns out that the component  $\pi_t^{(J)}$  simply captures the rolling mean and contributes very little to the total variance of  $x_t$  and has therefore little, if any, explanatory power for the original time series. Figure 2 provides graphical evidence supporting this statement for the case of consumption growth. The top panel plots the demeaned consumption growth together with the sum of its components, excluding the permanent one. The two series are close to each other with a correlation of 0.97. The bottom panel shows instead the difference between the unconditional mean of consumption growth and  $\pi_t^{(J)}$ . This difference vanishes as the sample length increases.<sup>13</sup> A similar conclusion can be drawn for the dividend growth and the financial ratios series. This is why in the rest of the paper we focus on demeaned time series which allows us to neglect the component  $\pi_t^{(J)}$  in both (6) and (7).

[Insert Figure 2 about here.]

We now turn to our asset pricing model in which all the time series of economic significance are described as sum of components with different persistence levels, as in equations (6) or, alternatively, (7).

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<sup>12</sup>In OTT2010b we show that, for the case where the series of observations is infinite and if the time series had a component which persists beyond any time scale, for instance if the time series were I(1), one would have that  $\pi_t^\infty$  would converge to the Beveridge-Nelson permanent component. Intuitively one could in principle keep on iterating over (4) and (5). However an arbitrary number of iterations of the moving average filter on the integrated time series will not be sufficient to remove such a permanent component.

<sup>13</sup>Intuitively equation (4) tells us that  $\pi_t^{(J)}$  simply captures the rolling mean, which, for any stationary time series, asymptotically converges to the sample mean, i.e.  $\pi_t^{(J)} \approx E[x_t]$  for  $t$  large enough.

## B.2. The Long-run Risk Model with Heterogeneous Persistence

Following the approach discussed in the previous section, we incorporate in the standard long-run risk model the decomposition of time series into components with different levels of persistence so that the log consumption growth,  $g_t$ , and the log dividend growth,  $gd_t$  take the following form:

$$g_t = \sum_{j=1}^J g_{j,t}$$

$$gd_t = \sum_{j=1}^J gd_{j,t}$$

where  $g_{j,t}$  and  $gd_{j,t}$  denote the components with level of persistence  $j$  as defined in the previous section. The novelty now is to assume that each component of consumption growth,  $g_{j,t}$  and of dividend growth,  $gd_{j,t}$  is driven by its own state variable,  $x_{j,t}$ , i.e.

$$g_{j,t+2j} = x_{j,t} + e_{j,t+2j}^g \tag{8}$$

$$e_{j,t+2j}^g \sim N(0, \sigma_{g,j}^2)$$

$$gd_{j,t+2j} = \phi_j x_{j,t} + e_{j,t+2j}^d \tag{9}$$

$$e_{j,t+2j}^d \sim N(0, \sigma_{d,j}^2)$$

where we allow the shocks to be correlated across levels of persistence (for fixed time  $t$ ) but not across time (for fixed persistence level  $j$ ) and we assume the consumption shocks  $e_{j,t+2j}^g$  to be mutually independent from the dividend ones  $e_{j,t+2j}^d$ . To close the dynamics of the model we assume that the components  $\{x_{j,t}\}_{j=1}^J$  follow a multiscale autoregressive process, i.e.

$$x_{j,t+2j} = \rho_j x_{j,t} + \varepsilon_{j,t+2j} \tag{10}$$

$$\varepsilon_{j,t+2j} \sim N(0, (\sigma^{(j)})^2)$$

In words, we are modeling separately the conditional mean  $x_{j,t}$  of each one of the components of consumption and dividends growth. Importantly, equations (8) to (10) represent a natural way to incorporate persistence heterogeneity in the long-run risk framework while retaining its pedagogical simplicity. On one side, in fact, these equations allow consumption growth to accommodate different degrees of persistence so to break the link between the autocorrelation of consumption and price-dividend. On the other side they maintain the simplicity of having, at each level of persistence  $j$ , only one variable driving the respective

consumption component.

To give economic and structural meaning to the parameters we assume, as usual, a pure exchange economy with a representative agent with Epstein-Zin recursive preferences. The well known Euler condition for such an agent is:

$$E_t \left[ e^{m_{t+1} + r_{t+1}^i} \right] = 1 \quad (11)$$

where  $m_{t+1}$  is the log stochastic discount factor given by

$$m_{t+1} = \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{t+1}^a, \quad (12)$$

$r_{t+1}^a$  is the log return of the claim which distributes a dividend equals to aggregate consumption and  $r_{t+1}^i$  is the log return on any asset  $i$ . The parameter  $\beta$  is the preference discount factor. The preference parameter  $\psi$  measures the intertemporal elasticity of substitution,  $\gamma$  measures the risk aversion and  $\theta = (1 - \gamma) / (1 - 1/\psi)$ .

In what follows we provide the basic steps to determine the pricing kernel and risk premia on the market portfolio in our long-run risk model with persistence heterogeneity.<sup>14</sup> Recall first that by the standard Campbell and Shiller (1988) log-linear approximation for returns one obtains:

$$\begin{aligned} r_{a,t+1} &= \kappa_0 + \kappa_1 z_{t+1}^a - z_t^a + g_{t+1} \\ r_{m,t+1} &= \kappa_{0,m} + \kappa_{1,m} z_{t+1}^m - z_t^m + g d_{t+1} \end{aligned} \quad (13)$$

where  $z_t^a$ ,  $z_t^m$ , denote the log price-consumption and the log price-dividend ratio respectively. Recalling our decomposition of consumption and dividends into components with different levels of persistence, and denoting with  $z_{j,t}^a$ ,  $z_{j,t}^m$ , the components with persistence  $j$  of the (log) price-consumption ratio and (log) price-dividend ratio respectively, it is natural to conjecture that there exists *component by component* a linear relation between the financial ratios and our state variables  $x_{j,t}$ , i.e.

$$\begin{aligned} z_{j,t}^a &= A_{0,j} + A_j x_{j,t} \\ z_{j,t}^m &= A_{0,j}^m + A_j^m x_{j,t} \end{aligned} \quad (14)$$

As long as  $A_j$  and  $A_j^m$  are not vanishing, these relations and equation (8) together imply that the components of price-consumption  $z_{j,t}^a$  and price-dividends  $z_{j,t}^m$  lead the component

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<sup>14</sup>All details behind our calculations are given in Appendix B.

of consumption and dividends with the same level of persistence  $j$ .

The values of  $A_{0,j}, A_j, A_{0,j}^m, A_j^m$  in terms of the parameters of the model are obtained from the Euler condition (11) after the log stochastic discount factors and the returns are all expressed in terms of the factors  $\{x_{j,t}\}_{j=1}^J$  and of the innovations  $\{e_{j,t+2j}^g\}_j$  and  $\{\varepsilon_{j,t+2j}\}_j$ . In Appendix B we show that plugging these expressions for the stochastic discount factor and for the returns into the Euler equation and using the method of undetermined coefficients one obtains a system of equations for the coefficients  $A_{0,j}, A_j, A_{0,j}^m, A_j^m$ , the solution of which is given by the following vectors of sensitivities:

$$\begin{aligned}\underline{A} &= \left(1 - \frac{1}{\psi}\right) (\mathbb{I}_J - \kappa_1 M)^{-1} \underline{1} \\ \underline{A}_m &= (\mathbb{I}_J - \kappa_{1,m} M)^{-1} \left(\underline{\phi} - \frac{1}{\psi} \underline{1}\right)\end{aligned}$$

where

$$M = \text{diag}(\rho_1, \dots, \rho_J)$$

i.e. the matrix collecting on the diagonal the persistence parameters of the components,  $\mathbb{I}_J$  is the identity matrix,  $\underline{\phi}$  is a column vector with entries  $\phi_1, \dots, \phi_J$  which reflect the exposures of the market dividends components to the consumption growth ones and  $\underline{A}$  and  $\underline{A}_m$  denote the column vectors with entries,  $A_1, \dots, A_J, A_1^m, \dots, A_J^m$ , respectively.

To study the implications of persistence heterogeneity for the equity premium recall that the risk premium on any asset  $i$  satisfies, in this set-up,  $E_t[r_{i,t+1} - r_{f,t}] + 0.5\sigma_{r_{i,t}}^2 = -\text{cov}_t(m_{t+1}, r_{i,t+1})$ . In the Appendix we show that the innovations of the stochastic discount factor are given by

$$m_{t+1} - E_t[m_{t+1}] = -\left(\frac{\theta}{\psi} - \theta + 1\right) e_{t+1}^g - \kappa_1(1 - \theta)\underline{A} \cdot (\varepsilon_{t+1}) \quad (15)$$

while analogous steps yield the following expressions for the return innovations

$$\begin{aligned}r_{a,t+1} - E_t[r_{a,t+1}] &= e_{t+1}^g + \kappa_1 \underline{A} \cdot \varepsilon_{t+1} \\ r_{m,t+1} - E_t[r_{m,t+1}] &= e_{t+1}^d + \kappa_{1,m} \underline{A}_m \cdot \varepsilon_{t+1}\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\varepsilon}_{t+1}^\top &\equiv [\varepsilon_{1,t+2^1}, \dots, \varepsilon_{J,t+2^J}] \\ e_{t+1}^g &\equiv \sum_j e_{j,t+2^j}^g \\ e_{t+1}^d &\equiv \sum_j e_{j,t+2^j}^d\end{aligned}$$

With the innovations to the equilibrium returns at hand and using (15), one can finally compute the risk premia for the consumption claim asset,  $r_{a,t+1}$  and for the market portfolio,  $r_{m,t+1}$ , hence obtaining

$$\begin{aligned}E_t[r_{a,t+1} - r_{f,t}] + 0.5\sigma_{r_{a,t}}^2 &= \lambda_g \sigma_g^2 + \kappa_1 \underline{\lambda}_\varepsilon' \mathbf{Q} \underline{A} \\ E_t[r_{m,t+1} - r_{f,t}] + 0.5\sigma_{r_{m,t}}^2 &= \kappa_{1,m} \underline{\lambda}_\varepsilon' \mathbf{Q} \underline{A}_m\end{aligned}$$

where

$$\begin{aligned}\lambda_g &\equiv \left( \frac{\theta}{\psi} - \theta + 1 \right) \\ \underline{\lambda}_\varepsilon &\equiv \kappa_1 (1 - \theta) \underline{A} \\ \mathbf{Q} &= \mathbf{E}_t [\boldsymbol{\varepsilon}_{t+1} \boldsymbol{\varepsilon}_{t+1}'] \\ \sigma_g^2 &= \text{Var} (e_{t+1}^g)\end{aligned}$$

The innovations  $\varepsilon_{j,t+2^j}$  driving the components of consumption growth at scale  $j$  reflect the impact of current uncertainty many years into the future. In analogy with the interest rate literature, they are similar to forward rates and we therefore refer to  $\boldsymbol{\varepsilon}_{t+1}$  as the term structure of risk. The parameter  $\underline{\lambda}_\varepsilon$  determines the risk compensation for these innovations.<sup>15</sup> Whereas BY04 price only the shock with persistence 0.979 affecting the *aggregate* consumption growth, in our asset pricing model we price all the shocks, each characterized by its own level of persistence, driving the *components* of consumption growth. The expressions for  $\underline{\lambda}_\varepsilon$  reveal that the degrees of persistence of consumption growth components affect the risk premium on the asset. In particular a rise in persistence increases  $\underline{\lambda}_\varepsilon$ . The exposure of the market return to these shocks is  $\mathbf{Q} \underline{A}_m$ . Importantly the exposure of the market return is determined simultaneously by the size of the shocks as measured by their instantaneous volatility, captured by  $\mathbf{Q}$  and by their persistence as measured by their half life. It is therefore

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<sup>15</sup>The parameter  $\lambda_g$  determine the risk compensation for the independent consumption shock  $e_{t+1}^g$  is present also in BY04 model and it is standard as  $\lambda_g$  equals the risk aversion parameter  $\gamma$ .

key in order to obtain the entire term structure of risk-return trade-offs to decompose the aggregate shocks that impinge an economy along these two competing dimensions. The process defined in equation (10) is very well suited to capture this fact since it classifies the shocks  $\varepsilon_{j,t+2j}$  both by its volatility  $\sigma^{(j)}$  and its half-life as measured by  $\rho_j$ . This classification is key for our model because it allows to determine precisely the short-run and long run-dynamics. In fact although highly volatile shocks with low level of persistence  $j$  can dominate in the short-run, as the valuation horizon increases, the effect of these shocks is averaged out while persistent (high  $j$ ) trends emerge and play a pivotal role. Thus we have obtained a picture where the asset valuation involves the full term structure of shocks driving the predictable components of consumption growth. In the next section we empirically investigate some of the implications of our asset pricing model with heterogeneous persistence.

### III. Empirical implications

In this section we first analyze the statistical behavior of the components with different levels of persistence that can be filtered out of the relevant time series. We then use the time series disaggregated across levels of persistence to revisit the consumption predictability test proposed by Beeler and Campbell (2009) and to estimate the intertemporal elasticity of substitution (IES). Our main empirical finding can be summarized as follows: once consumption growth is disaggregated across different levels of persistence, consistently with our model, some of its components are predictable by financial ratios and the estimates of IES is found to be greater than one. Last but not least, setting risk aversion to the conservative value of  $\gamma = 5$ , we compute the term structure of risk premia implied by our model and we show that thin predictable components contribute significantly to the equity premium.

#### A. Persistence Based Decomposition of the Relevant Time Series

In order to evaluate the implications of the long-run risks model with persistence heterogeneity we look at four variables: the changes in log consumption and dividends, the log price-dividend ratio and the log price-consumption ratio. Following BY04 and Beeler and Campbell (2009)<sup>16</sup> we use data on US nondurables and services consumption from the Bureau of Economic Analysis. We make the standard “end-of-period” timing assumption that consumption during period  $t$  takes place at the end of the period. The price-dividend ratio and dividend growth rates are obtained from the CRSP files. All nominal quantities are converted to real, using the personal consumption deflator. We consider a postwar quarterly US series over the period 1947:Q2-2009:Q4 and for robustness a long-run annual series over

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<sup>16</sup>We thank Jason Beeler for kindly providing us with the data.



the period 1930-2009. In what follows we report results only for the consumption growth and the price-dividend ratio series. The conclusions for the dividend process are quite in line with that of the consumption series, while the price-consumption ratio behaves in quite the same way as the price-dividend series.

Since in Section II.A. (recall in particular in Figure 1) we have already commented upon the very different statistical behavior of the consumption growth and price-dividend ratio, we now turn to the analysis of the components of these two time series. To do so we apply the decomposition described in Section II.B.1. to the aggregate time series. Figure 3 and Figure 4 display the components of the consumption growth and the price-dividend ratio for the quarterly and annual sample respectively.

[Insert Figures 3, 4 about here.]

As we move toward higher persistence levels, i.e. greater  $j$ , a common long-run behavior between the two series becomes apparent. This fact will be further explored in the next section when we discuss in details the predictability of consumption and dividend growth components by the financial ratios.

To further dig into the statistical properties of the components of the time series used in our empirical analysis, we focus on three key dimensions: the unconditional correlation, the half-life or persistence and, finally, the contribution of each component to the total unconditional variance of the aggregate series. First we inspect the pairwise correlation between the components of consumption growth and report Pearson's p-values<sup>17</sup> in Table 2. The Pearson's correlation test indicates that almost all of the consumption growth components are pairwise uncorrelated. The correlation is significant at standard levels only between the second and third and between the third and fourth components. For these components, however, the Pearson's correlation coefficients are no greater than 0.15. Overall these findings suggest that, although the components with different periodicity extracted with our approach could be in principle correlated across different levels of persistence, in practice we find very little evidence in favor of interaction across levels.<sup>18</sup>

Second we show that each component is stationary and has a well defined (in terms of interval) level of persistence. In order to do so we fit to each component an autoregressive process of order one. Estimates of the autoregressive coefficients and the  $R^2$  are shown in

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<sup>17</sup>Similar results are obtained using Spearman rank-order correlation coefficients test.

<sup>18</sup>We carry exactly the same analysis for the dividend growth and the financial ratios. Analogous results are obtained; in fact the correlations are significant only at similar low levels of persistence and are never greater than 0.15 for all the time series. Although we do not report them here they are available upon request.

Table 4 and 5 for the consumption process and price-dividend ratio, respectively. We verify that each component is strongly stationary and has a degree of persistence identified by the root of the autoregressive process.

[Insert Tables 2, 3, 4 and 5 about here.]

Now recall that the filtering procedure described in Section **II.B.1.** should identify, at level of persistence  $j$ , the fluctuations of the generic time series with periodicity belonging to the interval  $[2^{j-1}, 2^j)$ . To check the goodness of our method we denote with  $\rho_j$  the root of the autoregressive process of order one fitted onto the  $j$ -th component of the time series<sup>19</sup> and approximate the half-life  $HL(j)$  of this component by the standard relation  $HL(j) \simeq -\ln(2)/\ln(\rho_j)$ . Then consistent with our discussion in Section **II.B.1.** we should observe that the so computed half-life  $HL(j)$  for the generic component at level of persistence  $j$  is included in the interval  $[2^{j-1}, 2^j)$ . This is indeed the case and in particular we find that the half-life extrapolated from the autoregressive root is generally very close to the lower bound of the interval.<sup>20</sup>

Finally we report in Table 3 the contribution of the components  $g_{j,t}$  and  $z_{j,t}^m$  to the total variance of consumption growth and price-dividend respectively. We note that the highly persistent components of consumption growth yield a minor contribution to the total variance of the aggregate series. In particular each of the sixth and seventh components yield about 5% of total variance. The opposite happens for the price-dividend series: the components at levels 6 and 7 account for more than half of the total variance. This evidence contributes to explain why the aggregate time series of consumption and price-dividend have a very different persistence behavior. In fact, since the great part of the variability in consumption and price-dividend is explained by high frequency (i.e. low  $j$ ) and low frequency (i.e. high  $j$ ) components respectively, then the aggregate time series of consumption growth and price-dividend will resembles a white noise and a (close to) unit root process, respectively. The fact that the highly persistent components contribute for a very small fraction to the total volatility of aggregate consumption growth explains also why the predictability that exists at a specific level of persistence  $j$  disappears at the aggregate level. Indeed since the long-run components of consumption at level of persistence  $j = 6$  and  $j = 7$  are clearly overwhelmed by the high frequency noise then the comovements highlighted in Figure 3

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<sup>19</sup>The autoregressive coefficients  $\rho_j$  for the consumption growth and the price-dividend ratio series are reported in Tables 4 and 5 respectively.

<sup>20</sup>For instance the half-life of the consumption growth components at levels  $j = 5, 6, 7$  is about 5, 11 and 21 years, respectively. This numbers are consistent with the fact that these components should identify fluctuations with periodicity, measured in years, belonging to the interval  $[4, 8)$ ,  $[8, 16)$  and  $[16, 32)$  respectively.

between consumption growth and price-dividend ratio at these levels of persistence do not emerge unless we suitably separate the informative low frequency components from the noisy high frequency ones. We will make this argument more formal in subsection *C*.

The above discussion and the evidence presented so far suggest therefore that the time series of interest in this paper can empirically be well represented as the sum of autoregressive components, each of which has an half-life belonging to a well defined interval. Intuitively this makes sense from an economic point of view because, for instance, consumption data results from an aggregation through time and across heterogeneous households. Even assuming that each component in this aggregation follows a simple autoregressive process, still the aggregation procedure would generate persistence heterogeneity.<sup>21</sup> Finally this section, and in particular Figures 3 and 4, confirm the importance of decomposing across levels of persistence the time series of interest in order to uncover important economic relations that would be otherwise hindered by the noisy components. In the next subsections we employ the disaggregated time series in order to test whether the empirical implications of our model are indeed supported by the data.

### *B. Predictability of Consumption and Dividend Growth*

In the standard long-run risk model stock prices respond strongly to variation in expected future aggregate consumption growth. The innovation we propose in this paper consists in the fact that this relation holds componentwise even when it is not necessarily required to hold for the aggregate series.

In particular our long-run risk model with persistence heterogeneity implies that the components of price-consumption  $z_{j,t}^a$  and price-dividends  $z_{j,t}^m$  lead the components of consumption with the same level of persistence  $j$  (see relations (8) and (14)). In light of our model, therefore, we first disaggregate our variables across different levels of persistence and then we quantify the predictability at each level of persistence by running the following regressions:

$$\begin{aligned} g_{j,t+2j} &= \beta_{0,j} + \beta_{1,j}^g z_{j,t}^m + \varepsilon_{t+2j,j} \\ g_{j,t+2j} &= \beta_{0,j} + \beta_{1,j}^a z_{j,t}^a + \varepsilon_{t+2j,j} \end{aligned} \tag{16}$$

Although these regressions resemble those run by Beeler and Campbell (2009) we remark that while they focus on the aggregate time series we study instead predictability at a spe-

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<sup>21</sup>Aggregating these heterogeneous components results in the long memory property of aggregate consumption which is well documented in the literature (see e.g. Lippi and Zaffaroni (1998) and Thornton (2008)).

cific level of persistence. Clearly from an empirical point of view it can be the case that the predictability relation holds for all, some of or none of the persistence levels  $j$ . It is notationally useful to let  $S \subset \{1, \dots, J\}$  denote the set of persistence levels for which the components of consumption growth and dividend growth are led by the financial ratios. Hence the persistence levels belonging to  $S$  select the characteristic components of consumption growth which are predictable. Results for the quarterly sample are reported in Table 6 and Table 7. Table 6 shows that at levels of persistence  $j = 3, 6, 7$  the coefficients on the price-dividend ratio are statistically significant at the 5% level and that the sixth and seventh components account for a great part of the variation in the future consumption growth at the corresponding scale, the  $R^2$  being between 24% and 38% respectively. Empirically we have  $S = \{3, 6, 7\}$ , i.e. the components of the price-dividend ratio that actually lead the corresponding components of consumption growth have cycles of length, measured in years, belonging to the intervals  $\{[1, 2], [8, 16], [16, 32]\}$ . Table 7 shows that the same components of consumption growth that are predictable by the price-dividend ratio are also predictable by the price-consumption ratio. Finally, as a robustness check we perform the same consumption predictability test but now using annual data. Table 8 reports the results and shows that they are consistent with the ones obtained for the quarterly series: the component with level of persistence  $j = 3, 6, 7$  turn out to be the only statistically significant ones.

[Insert Tables 6, 7 and 8 about here.]

Our long-run risk model with heterogeneous levels of persistence implies also that the same latent state variables  $x_{j,t}$  that generate persistent variations in consumption growth at levels of persistence  $j \in S$  should generate variations at the same levels of persistence in the dividend growth components. A natural test of this implication is to see if the components of price-consumption  $z_{j,t}^a$  and price-dividends  $z_{j,t}^m$  lead the corresponding dividend growth components. Table 9 reports the results<sup>22</sup> and shows that the very same components at levels 3, 6, 7 that are significant for consumption growth are also statistically significant at the 5% level for the dividend growth. The sixth and seventh components, in particular, account for a great part of the variation in the expected future dividend growth at the corresponding scale, the  $R^2$  being between 25% and 38% respectively.

[Insert Table 9 about here.]

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<sup>22</sup>We report results only for the case where the regressor is the price-dividend ratio and the data are quarterly. Conclusions do not change when we use the price-consumption series as the regressor and/or when we use longer annual series.

Once again we remark that we are not providing evidence in favor of strong predictability of aggregate consumption growth but, in fact, we just provide evidence for the presence of persistent components in consumption which cause at the corresponding levels of persistence, stock price variability (relative to dividends). Therefore the above results are not in contrast with the ones of Beeler and Campbell (2009). The persistent variations in consumption growth, unless filtered, are indeed overwhelmed by measurement error and aggregate predictability therefore does not occur.

It is now interesting to compare the filtering and estimation approach used in this paper with two other techniques, namely cointegration and long-horizon regressions, which have been adopted to analyze long-run economic and financial relations.

The cointegrated approach has been used in recent empirical work, e.g. Bansal et al. (2007) and Ferson, Nallareddy and Xie (2010), to model the persistent component in consumption and dividend processes. In particular Bansal et al. (2007) argue that the cointegrating relation between dividends and consumption is a good measure of long-run consumption risks. Our long-run risk model with heterogeneity in persistence does not rule out this possibility. In fact the presence in consumption and dividends series of seasonal patterns with a very long half-life suggests to interpret the components at levels 6 and 7 as the common trend driving the cointegrating relation between consumption and dividends. However our interpretation points to a kind of cointegration different from the traditional one used in Bansal et al. (2007) and Ferson et al. (2010), since our components are persistent but not permanent. Therefore our suggested cointegration should rely upon the notion of seasonal cointegration relationships (see Osborn (1993)), which is a generalization of the classic concept of cointegration proposed in the seminal work of Engle and Granger (1987) where we may consider cointegrating relations not only at zero frequency but also at other (two) frequencies connected with long-run cycles (Engle et al., 1993).<sup>23</sup> In this framework the parallel common movements in the components at persistence levels 6 and 7 in the dividends and consumption variables may be therefore interpreted as a cointegration at seasonal frequencies and long-run risk may well be captured by (seasonal) cointegration relations.

With regard to long-horizon regressions, we note that our approach, which uses filtered regressand and regressors in ordinary least squares, relates to the one of Torous et al. (2004)<sup>24</sup>

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<sup>23</sup>According to the definition proposed primary by Hylleberg et al. (1990) seasonal cointegration means the cointegration only at seasonal frequencies. However in many theoretical and practical works (Johansen and Schaumburg, 1999), the term seasonal cointegration analysis corresponds to the cointegration analysis conducted at not only seasonal but nonseasonal frequencies as well. We will use this term in this more general meaning.

<sup>24</sup>In fact Torous et al. (2004) shows that the OLS estimator is consistent when both the regressor and the regressand are aggregated over non-overlapping periods (cases 2 and 4 in their paper), i.e., regressing a

and Bandi and Perron (2008) where the authors study predictability relations aggregating both the regressor and regressand over non-overlapping periods having the same length. This is very similar to what we do in equations (16) where we regress the consumption components obtained using aggregate consumption data in the time window  $t + 1$  to  $t + 2^j$  onto the price-dividend components in the time window  $t - 2^j$  to  $t$ .

From the methodological point of view, the same statistical caveats which are present in long-horizon predictive regressions also apply here. In particular Section **II.B.1.** explains that our filtering procedure is based on moving-average filters. This smoothing operation in turns generates autocorrelation in the data. This is a critical issue for the OLS procedure since both the regressor and the regressand become highly persistent and imposes the use of properly modified statistical significance indicators. To address this problem in this paper the standard errors are computed using the method proposed by Hansen and Hodrick (1980) to correct for serially correlated errors.<sup>25</sup>

Finally since our entire procedure relies on the linear filtering technique described in Section **II.B.1.**, to decompose our time series, it is important to check whether our results are driven by the particular choice of filter made here. As a robustness check for the filtering procedure we use the bandpass filter described in Christiano and Fitzgerald (2003).<sup>26</sup> The choice of frequencies interval characterizing the band-pass filter is determined according to Section **II.B.1.**: we band-pass the consumption growth and price-dividend ratio over the interval  $\left[\frac{f_{max}}{2^j}, \frac{f_{max}}{2^{j-1}}\right)$   $j = 1, \dots, 8$ . Results are reported in Table 10. We find again evidence of a very long-lasting component in consumption growth (note in fact that the  $R^2$  spikes at the scales 6 and 7).

[Insert Table 10 about here.]

In summary, we find that at suitably defined levels of persistence stock prices predict the long-run prospects for consumption and dividend growth. In the next section we reconcile our evidence of componentwise predictability with the one presented in the very recent long-run risk literature which rules out predictability at the aggregate level.

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long-horizon variable against the other.

<sup>25</sup>Results are robust when standard errors are computed using Newey-West with optimal lag length estimators, similar to Beeler and Campbell (2009).

<sup>26</sup>Results are practically identical under Baxter and King's (1999) band-pass filter.

### C. *Componentwise versus Aggregate Consumption Predictability and the Errors-in-Variable Problem*

Why does the componentwise predictability, i.e. the predictability we find when we disaggregate the consumption and dividend growth across different levels of persistence, wash away at the aggregate level? The main reason is that if one does not filter appropriately the time series of interest then the empirical results are plagued by an errors-in-variables problem which hides the predictability of components with a specific levels of persistence.

To understand where this errors-in-variables problem arises, consider the following simple case where consumption growth is the sum of a persistent component  $g_t^*$ , and an idiosyncratic one, and the price-dividend ratio is a noisy proxy for the state variable  $x_t$  driving the persistent component of consumption growth, i.e.

$$\begin{aligned} g_{t+1}^* &= \mu + x_t + \sigma\eta_{t+1} \\ g_t &= g_t^* + \nu_{2,t} \\ z_t^m &= A_{0,m} + A_{1,m}x_t + \nu_{1,t} \end{aligned} \tag{17}$$

where  $\nu_{1,t}$  and  $\nu_{2,t}$  are the consumption growth and price-dividend ratio idiosyncratic components, respectively. Note that this model differs from the standard long-run risk one in two dimensions. First, the relation (1) does not hold for the aggregate consumption but only for its persistent component  $g_t^*$ . Second, the key aspect of our simple model is that the components  $\nu_{1,t}, \nu_{2,t}$  and the common component  $x_t$  across the two processes, have different levels of persistence. Similarly to the standard long-run risk model we assume that  $g_t^*$  is highly persistent and explains a small fraction of the total variance of the consumption growth.

If one now looks for predictability at the aggregate level and runs the following regression:

$$g_{t+1} = \beta_0 + \beta_1 z_t^m + \epsilon_{t+1}$$

the predictive effect could be largely underestimated. In fact notice that the price-dividend ratio  $z_t^m$  covaries only with the small component  $g_t^*$  of consumption growth. This covariation manifests itself on a characteristic time scale much longer than the interval of observation and thus it could be hidden at short horizons by the large in volatility idiosyncratic component.

More formally the results are plagued by a typical errors-in-variable problem,<sup>27</sup> the OLS estimator being downward biased and inconsistent. In order to understand why this happens,

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<sup>27</sup>In the standard EIV set-up  $\nu_{1,t}$  and  $\nu_{2,t}$  are i.i.d homoskedastic shocks.

let's substitute  $g_t^* = g_t - \nu_{2,t}$  and  $x_t = \frac{z_t^m - \nu_{1,t} - A_{0,m}}{A_{1,m}}$  into the first equation in (17). This yields

$$\begin{aligned} g_{t+1} &= \mu + \frac{z_t^m - \nu_{1,t} - A_{0,m}}{A_{1,m}} + \sigma\eta_{t+1} + \nu_{2,t+1} \\ &= \beta_0 + \beta_1 z_t^m + \underbrace{\sigma\eta_{t+1} + \nu_{2,t+1} - \beta_1 \nu_{1,t}}_{\varepsilon_{t+1}} \end{aligned} \quad (18)$$

It is now easy to see that the presence of measurement error in  $g_t$  and  $z_t^m$  leads to an increase in the variance of the error term. In addition the regressor  $z_t^m$  and the error term  $\varepsilon_{t+1}$  in (18) are correlated and this covariance does not depend on the sample size, it does not vanish asymptotically, and hence the OLS estimator is downward biased and inconsistent. These facts contribute to explain why a simple regression on scaled stock prices does not reveal the long-run prospects for consumption.

Hence when the underlying economic relation holds only for the unobserved processes  $g_t^*$  and  $x_t$  and when the size of the component  $g_t^*$  measured by instantaneous (single period) volatility is very small compared to the instantaneous total volatility of consumption growth as in this example, the filtering of these latent components becomes a critical issue for the empirical analysis.<sup>28</sup> Our approach manages to uncover the true economic relation by pre-filtering out the idiosyncratic components  $\nu_{1,t}$  and  $\nu_{2,t}$  from the observables series  $g_t$  and  $z_t^m$  and then by using the filtered regressand  $g_t^*$  and regressor  $x_t$  in a simple OLS framework. The pre-filtering procedure exploits the key assumption that the idiosyncratic components and the predictable one have different levels of persistence.

#### D. Identification of the Consumption Components

In the previous section we highlighted the presence of consumption components predictability at specific levels of persistence, namely  $j \in S = \{3, 6, 7\}$ . It is interesting to investigate the existence of reasonable economic proxies for these consumption components. To search for these proxies we rely on time series that are economically significant, that are characterized by an half-life close to the one of the components they are to proxy for, and that are significantly correlated with such component.

First we focus on the third component filtered out of consumption growth,  $g_{3,t}$ , whose mean reversion is between one and two years. In order to identify this component with observable economic factors we follow the lead of Jagannathan and Wang (2007) and Moller and Rangvid (2010) who analyze the ability of the fourth-quarter consumption growth rate

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<sup>28</sup>The importance of filtering procedures to avoid EIV problem has been recently highlighted, see also Gencay and Gradojevic (2009).



to predict expected excess returns on stocks. Importantly, this variable aims at capturing economic and financial choices happening with yearly frequency. Figure 5 reports the series  $g_{3,t}$  and the one used by Moller and Rangvid (2010). Quite remarkably the correlation between these two series is 0.60 in our sample period.

[Insert Figure 5 about here.]

The economic idea behind Jagannathan and Wang (2007) and Moller and Rangvid (2010) is motivated by an alignment between consumption and investment decisions in the fourth quarter. Indeed the infrequent points in time where investors decide to review their investments are most likely influenced by culture (such as Christmas) and institutional features (such as end-of-year bonuses and the tax consequences of capital gains and losses, which both occur mainly in the fourth quarter of the year). If the points in time when consumption and investment decisions are taken coincide, such as in the fourth quarter, a clear relation between consumption decisions and stock prices should emerge. Our third component seems to be a good candidate to capture this effect.

We now turn to the sixth and seventh components, which are slow moving series with an half life of about 8 and 16 years, respectively. To search for a valid proxy for the sixth component we follow the lead of the recent macro-finance literature suggesting that technology prospects should be positively related with aggregate consumption. For instance Garleanu et al. (2009) argue that consumption growth over long-horizons should reveal the position of the economy with respect to the technological cycle. In particular the authors show that predictable components of consumption that occur at cycles between 10 and 15 years are due to the presence of large infrequent embodied technology shocks. Similarly Kaltenbrunner and Lochstoer (2010), Kung and Schmid (2011) and Croce (2010) investigate the implications of long-run risk in a general equilibrium production economy and show how shocks to productivity growth generate predictable movements in consumption growth. Moreover Hsu and Huang (2010) show that changes in technology prospects are risk factors which explain the growth of aggregate consumption. Based on this motivating evidence we investigate whether our consumption component at level of persistence  $j = 6$  plays the role of shocks to productivity growth. We plot in Figure 6 the sixth component of consumption growth together with the long-run multifactor productivity index.<sup>29</sup> The correlation between the two series is a comforting 0.64 giving further support to the idea that highly persistent time-variation in (components of) consumption growth, i.e. long-run risk, can reflect permanent technology

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<sup>29</sup>Data are from the Bureau of Labor Statistics; the sample spans 1948-2008. In particular, the index measures the value-added output per combined unit of labor and capital input in private business and private nonfarm business. Available at <ftp://ftp.bls.gov/pub/special.requests/opt/mp/prod3.mfptablehis.zip>.

shocks.

[Insert Figures 6 and 7 about here.]

As for the seventh component, we look at the literature linking demographic fluctuations to long-run stock prices. For instance, Geanakoplos, Magill and Quinzii (2004) and more recently Favero, Gozluklu and Tamoni (2010) have shown that changes in the distribution of the population age account for long-run cycles in the U.S. stock market. Geanakoplos et al. (2004) analyze an overlapping generation model in which the demographic structure mimics the pattern of live births in the US. Importantly live births in the US have featured alternating twenty-year periods of boom and busts, and therefore are consistent with the half-life of our seventh component. One implication of their model is that real stock prices are positively related to the ratio of “middle aged population to population of young adults” (the so called MY ratio) even when investors are forward-looking and rational. Favero, Gozluklu and Tamoni (2010) provide empirical evidence for the presence of a slowly evolving trend in the log dividend-price ratio, determined by the MY ratio. Figure 6 reports the seventh component of consumption growth along with the series for MY. The correlation in the full sample is equal to 0.44. Note that the demographic variable still leaves place for unexplained variability in the slow-moving component of consumption growth. A possible way to improve the identification of the seventh component would entail considering persistent improvements in the degree of risk sharing among households or regions (see Lustig and Van-Nieuwerburgh (2006)) or the persistent changes in the tax code (see McGrattan and Prescott (2005)).

Summing up we have shown that the predictable components of consumption growth are highly correlated with well known structural drivers of consumption variability. On the longest side we found demographic shocks to be highly correlated with the component describing consumption growth variations that occurs on time scales which range between 16 and 32 years. On the intermediate side, long-run productivity growth explains cyclical variations with time scale between 8 and 16 years. On the shortest side, finally, we find an high frequency predictable component with a yearly half-life that can be identified with the well documented fourth quarter effect.

### *E. The Risk-free Rate and Intertemporal Elasticity of Substitution*

In this Section we aim at estimating the intertemporal elasticity of substitution (IES) by separating the consumption growth into cyclical components. To obtain the empirical relation that allows us to estimate the IES we derive first the expression for the risk-free rate in our long-run risk model. This allows us to obtain, under the maintained assumption of non stochastic second moments, a simple linear relation that links the components of

the real interest rate to the ones of consumption growth via the intertemporal elasticity of substitution.

The starting point of this analysis is the following expression which is derived in Appendix C.:<sup>30</sup>

$$r_{f,t+1} = \frac{1}{\psi} \sum_j x_{j,t} \quad (19)$$

In order to bring this relation to the data we need two further steps. First we rewrite relation (19) by applying the forward decomposition (7) to its left-end side. By doing so we obtain

$$\sum_{j=1}^J r_{f,j,t+2j} = \frac{1}{\psi} \sum_j x_{j,t}$$

Then using the equation (8) to link the latent component  $x_{j,t}$  with the consumption growth component and introducing measurement errors, the following set of  $J$  testable implications obtains:<sup>31</sup>

$$r_{f,j,t+2j} = \frac{1}{\psi} g_{j,t+2j} + \sigma_{f,j} \eta_{j,t+2j} \quad j = 1, \dots, J \quad (20)$$

This is a set of relations which is constrained by the condition that the coefficient linking in the linear regression the information content of the risk-free rate components to the consumption growth ones, must be the same at all levels of persistence. Table 11 displays the results of the above constrained system of equations. The first row shows that when we decompose the risk-free rate and consumption growth across the different levels of persistence we estimate the IES to be significant at standard levels and equal to 4.762. The reported t-statistics are based on GMM corrected standard errors in order to cope with the overlapping observations problem introduced by our moving-averages. In this special case where the parameter to be estimated is equal at all levels of persistence, one can solve the overlapping problem by adopting the technique suggested in Fadili and Bullmore (2002). In particular Fadili and Bullmore (2002) suggest to (sub)sample the components at level of persistence  $j$  with frequency  $2^j$ <sup>32</sup> in order to get rid of the autocorrelation problem and then to apply

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<sup>30</sup>The relation in Appendix C. includes also a constant  $\alpha_f$ . As already said in Section II.B.1. we focus on demeaned time series which allows us to neglect  $\alpha_f$  and also the component  $\pi_t^{(J)}$  in both (6) and (7).

<sup>31</sup>Indeed using the equation (8) one obtains  $\sum_{j=1}^J r_{f,j,t+2j} = \frac{1}{\psi} \sum_j g_{j,t+2j} - \frac{1}{\psi} \sum_j e_{j,t+2j}^g$  and theoretically one could draw inference on  $\psi$  both from the loadings on the consumption growth components and from the variance of the innovations. However observe that once we add measurement errors at each time scale the volatilities of the innovations  $e_{j,t+2j}^g$  and the volatilities of the measurement errors cannot be separated on the basis of the information set that we have. We therefore draw inference based only on the loading coefficients on the consumption growth components.

<sup>32</sup>Note that if we apply our decomposition to a time series with  $T = 2^J$  elements we then obtain  $J$  components with  $T$  elements. If we subsample the components we obtain a new time series with  $T/2 + T/4 + \dots + T/2^J = T$  elements, that is the new sampled series has the same length of the original one.

to the so obtained sampled time series the generalized least squares estimator (GLS).<sup>33</sup> We report the results obtained using this approach in the second and third rows of Table 11. The second row uses the sample period 1978Q1-2009Q4, that is exactly 128 data points. The third row uses the sample period 1948Q1-2009Q4, which is composed of 248 data points. and 8 data points are missing to reach the critical dimension of 256 points. We fill these 8 points with either a sequence of zeros or by using reflecting boundaries. The results are unaltered.<sup>34</sup> Importantly the estimates that we obtain are strongly significant, all above one and close to the value obtained in the case where overlapping data have been used.

[Insert Table 11 about here.]

It is useful to compare our results with the standard regression approach<sup>35</sup> originally suggested by Hansen and Singleton (1983) to estimate the elasticity of intertemporal substitution and followed by Hall (1988) and Campbell and Mankiw (1990), among many others. Observe that under the assumption that the conditional expectation of the consumption growth components, i.e. the latent variables  $\{x_{j,t}\}_{j=1}^J$ , sum up to the expectation of the aggregate consumption, i.e. the single latent variable  $x_t$  used in BY04, formally if  $\sum_j x_{j,t} = x_t$ , then equation (19) can be rewritten as

$$r_{f,t+1} = \frac{1}{\psi} x_t$$

and using the BY04 dynamics for consumption growth (see equation (1)) to proxy for the latent variable  $x_t$ , the set of relations (20) collapse to the standard regression<sup>36</sup>

$$r_{f,t+1} = \frac{1}{\psi} g_{t+1} + \sigma_f \eta_{t+1} \quad (21)$$

Empirical tests carried out by Hall (1988) and Campbell and Mankiw (1990) find a low estimate of  $\psi$  which contradicts the assumption of the long-run risk model that assets are

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<sup>33</sup>More precisely this estimator makes use of the decimated (not-redundant) Haar transform which yields a diagonalized covariance matrix of the regression errors, i.e. the off-diagonal elements can be set to zero. Diagonalization simplifies numerical identification of parameter estimates and implies that the WLS estimator is theoretically approximate to the best linear unbiased (BLU) estimator and can provide maximum likelihood estimates of both signal and noise parameters, namely  $\frac{1}{\psi}$  and  $\sigma_\eta$ .

<sup>34</sup>Fadili and Bullmore (2002) study in detail the effect of artifactual inter-coefficient correlations introduced by boundary correction at the limits of the data and show that WLS is unbiased over a wide range of data conditions and its efficiency closely approximates theoretically derived limits

<sup>35</sup>More in details instrumental variables (IV) are used.

<sup>36</sup>Alternatively, one can reverse the regression and estimate  $\Delta c_{t+1} = \beta_0 + \psi r_{f,t+1} + \eta_{t+1}$ . However, if  $\psi$  is large as it will turn out to be the case in our empirical exercise, then it is better to estimate the equation reported in the text.

priced by a representative agent with an elasticity of intertemporal substitution greater than one. A potential explanation of these results is that (particularly in postwar quarterly data) the real interest rate is very volatile relative to predictable variation in consumption growth (see Beeler and Campbell (2009) for a discussion). Therefore unless we disentangle the highly volatile noise component from the low volatile informative ones it would be difficult to estimate properly the IES.

This is exactly what we do when we apply the persistence based decomposition before running the regressions. In fact compared to regression (21), equation (20) mandates to consider all of the latent variables driving the consumption components. Thus the testable implications of our model require to properly take into account the heterogeneity in consumption growth generated by the mixture of highly volatile and the slowly evolving components (see again equation (19)). By doing so a robust estimate larger than one is obtained producing empirical support to a key hypothesis of the long-run risk valuation approach. Our empirical findings are in agreement with previous studies, e.g. Attanasio and Weber (1993), Beaudry and van Wincoop (1996) and Vissing-Jorgensen (2002) who find values for  $\psi$  higher than one. In fact similar to these studies we present empirical evidence on aggregation problems with the relationship between consumption growth and the real interest rate and we show that using disaggregated consumption data is key to find a value for the IES greater than one. However whereas these studies focus on consumption data disaggregated at cohort-level, state-level and household level, respectively, we suggest persistence heterogeneity as an additional key dimension along which consumption can be disaggregated.

It is important to observe that these studies use consumption data disaggregated at cohort-level, state-level and household level respectively.

#### *F. The term structure of equity market risk premium*

In subsection *B.* we find that stock prices reveal the long-run prospects for consumption and dividend growth once the persistence level is properly taken into account. The ultimate relevance of the predictability effects in the components of consumption and dividends is related to the ability of these “thin persistent effects” to generate sizeable risk premia. It is therefore crucial to quantify the contribution of the predictable components to risk premia within our long-run risk model with persistence heterogeneity.

Recall that in our model the equity premium for the market portfolio  $r_{m,t+1}$  satisfies:

$$\begin{aligned} E_t[r_{m,t+1} - r_{f,t}] + 0.5\sigma_{r_{m,t}}^2 &= \kappa_{1,m}\underline{\lambda}_\varepsilon \mathbf{Q} \underline{A}_m \\ &= \kappa_1 \underbrace{\kappa_1(1-\theta)\underline{A}}_{\underline{\lambda}_\varepsilon} \mathbf{Q} \underline{A}_m \end{aligned} \quad (22)$$

where

$$\begin{aligned} \underline{A} &= \left(1 - \frac{1}{\psi}\right) (\mathbb{I}_J - \kappa_1 M)^{-1} \underline{\mathbf{1}} \\ \underline{A}_m &= (\mathbb{I}_J - \kappa_{1,m} M)^{-1} \left(\underline{\phi} - \frac{1}{\psi} \underline{\mathbf{1}}\right) \end{aligned}$$

The aggregate risk premia reflect the risk exposures  $\mathbf{Q} \underline{A}_m$  and the risk prices  $\underline{\lambda}_\varepsilon$  at all levels of persistence. We are then able to characterize the contribution to the equity premium of the risk components at each different horizon. This is similar in spirit to Hansen and Scheinkman (2009) and Borovicka, Hansen, Hendricks and Scheinkman (2009) where the authors are interested in the entire term structure of risk prices. Moreover note that whereas the risk prices tend to increase with the level of persistence, the risk exposures do not since as we have already seen the highly persistent predictable components contribute for a very small fraction to the total volatility of aggregate consumption growth. This yields non trivial dynamics and in particular the equity premium does not need to increase with the level of persistence.

Aside from  $\psi$  which has been already taken care of in the previous section, to compute the equity premium we need an estimate for  $M$ ,  $\mathbf{Q}$  and  $\underline{\phi}$ . With these parameters at hand we can obtain the equity premium for calibrated values of  $\gamma$ .

The risk aversion parameter is set to  $\gamma = 5$  whereas according to the previous section  $E$ . we set  $\hat{\psi} = 4.76$ .<sup>37</sup> The persistence levels matrix  $M$  and the innovations' variance-covariance matrix  $\mathbf{Q}$  are obtained by fitting a vector autoregressive system to the price-dividend components where the matrix  $M$  is restricted to be diagonal. Finally we need an estimates of  $\underline{\phi}$ . Recalling the equations for the consumption growth dynamics (8) and the linear relations (14) for the financial ratios it is immediate to verify that the coefficient  $\hat{\beta}_{1,j}^g$  obtained by regressing componentwise the consumption growth on the price-dividend produces an estimate of the coefficient  $\frac{1}{A_{j,m}}$  (see Table 6). By the same token, using the equations for the dividend growth (9) and together with equations (14), we see that the

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<sup>37</sup>Our conclusions do not change significantly even when we use the conservative value  $\psi = 1.5$  suggested in BY04.

coefficient  $\hat{\beta}_{1,j}^{gd}$  estimated from the regression of the components of log dividend growth on the components of log price-dividend ratio yields an estimate of  $\frac{\phi_j}{A_{j,m}}$  (see Table 9). Therefore an indirect estimate of  $\phi_j$  is given by

$$\hat{\phi}_j = \frac{\hat{\beta}_{1,j}^{gd}}{\hat{\beta}_{1,j}^g}$$

With these parameter values at hand we compute the equity premium. Results are reported in Table 12. Notice that the equity premium can be zero at some scale and this can happen either because the price of risk is zero or because the risk exposure is zero. For the market portfolio using the results in Table 6 and Table 9 we observe that the risk exposure can be different from zero only at scale 3, 6 and 7. The unique parameter which is calibrated is the risk aversion coefficient, which is set to the reasonable value of  $\gamma = 5$ . Hence we can conclude that on the basis of our empirical findings, within our long-run risk model with persistence heterogeneity, long-run risk offers a plausible solution to the equity premium puzzle.

[Insert Table 12 about here.]

#### IV. Conclusions

The above considerations prove that a classification of shocks based on the persistence based decomposition improves the discriminatory power of empirical tests on long run risk valuation models getting rid of an errors-in-variable problem generated by the heterogeneity of persistence.

This paper shows that a long-run risk model, where the effects of persistence heterogeneity is properly taken into account offers a credible explanation to many empirical results which seemed to contradict the long run valuation picture. Our results clearly indicate that any systematic empirical test of a long-run risk model must classify shocks across two competing dimensions, their size as measured by volatility and their persistence as measured by their half life.

Our proposal, the use of a persistent based decomposition, seems to offer interesting developments. It has to be remarked that pros and cons of filtering procedures have been discussed in the macroeconomic literature (see for instance Christiano and Fitzgerald (2003) and Canova (1998)) where it has been observed that sometimes results are not robust with respect to different choices of the filtering criterion. In fact the decomposition procedure introduces an additional source of model risk, hence an uncertainty averse agent should take it into account while forming expectations. In our analysis we assumed that the representative agent is uncertainty indifferent leaving for future research the analysis of the effects of

ambiguity aversion on valuation.

There's a number of research questions which are left open by our research both on the methodological and on the empirical side. In our understanding a systematic analysis of the asymptotic large sample theory for the persistence based decomposition would offer a natural and general framework to analyze the term structure of risk-return trade-offs in asset valuation. On the empirical side it is clear that the next step to verify the credibility of the long-run risk with persistence heterogeneity is the extension of the analysis to bond prices and to state dependent volatility. Preliminary research on this side has produced promising results but unavoidably requires the extension of the model with the inclusion of inflation among priced risks.



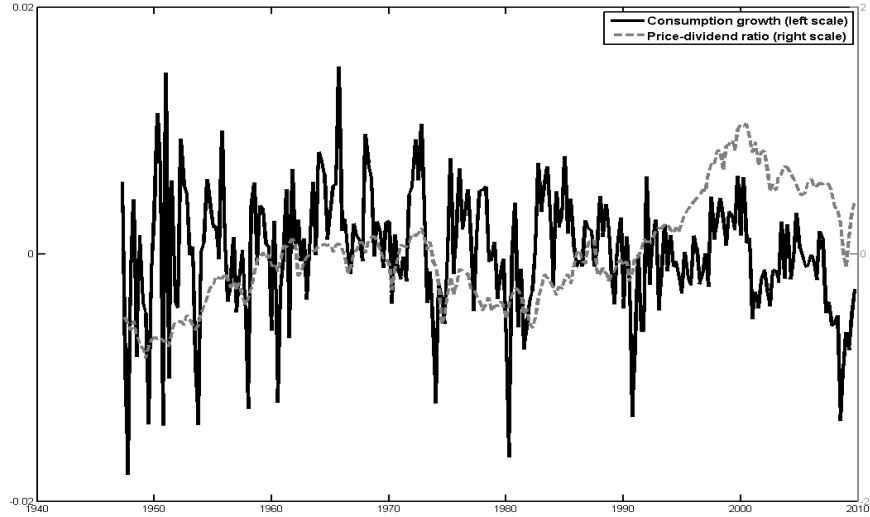


Figure 1: The figure displays the series of US consumption growth (nondurables and services) from the Bureau of Economic Analysis and the log price-dividend ratio (dashed line).

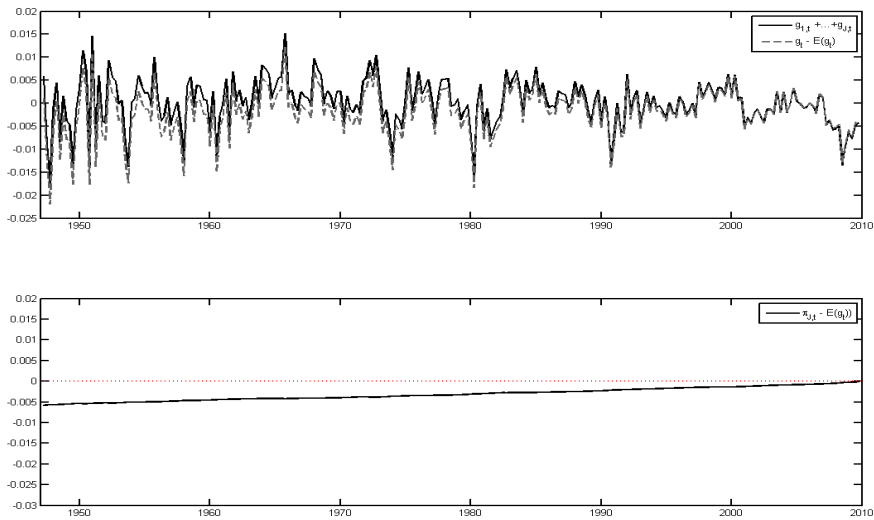


Figure 2: The figure displays the effect of removing the permanent component  $\pi_{J,t}$ . The top panel reports the *demeaned* series of US consumption growth (nondurables and services) from the Bureau of Economic Analysis consumption growth (dashed line) together with the sum of its components, excluding the permanent one (solid line). The correlation between the two series is 0.97. The bottom panel shows the difference between the permanent component  $\pi_{J,t}$  and the sample mean of the consumption growth. This difference vanishes as the sample length increases. Both panels are obtained for  $J = 8$ .

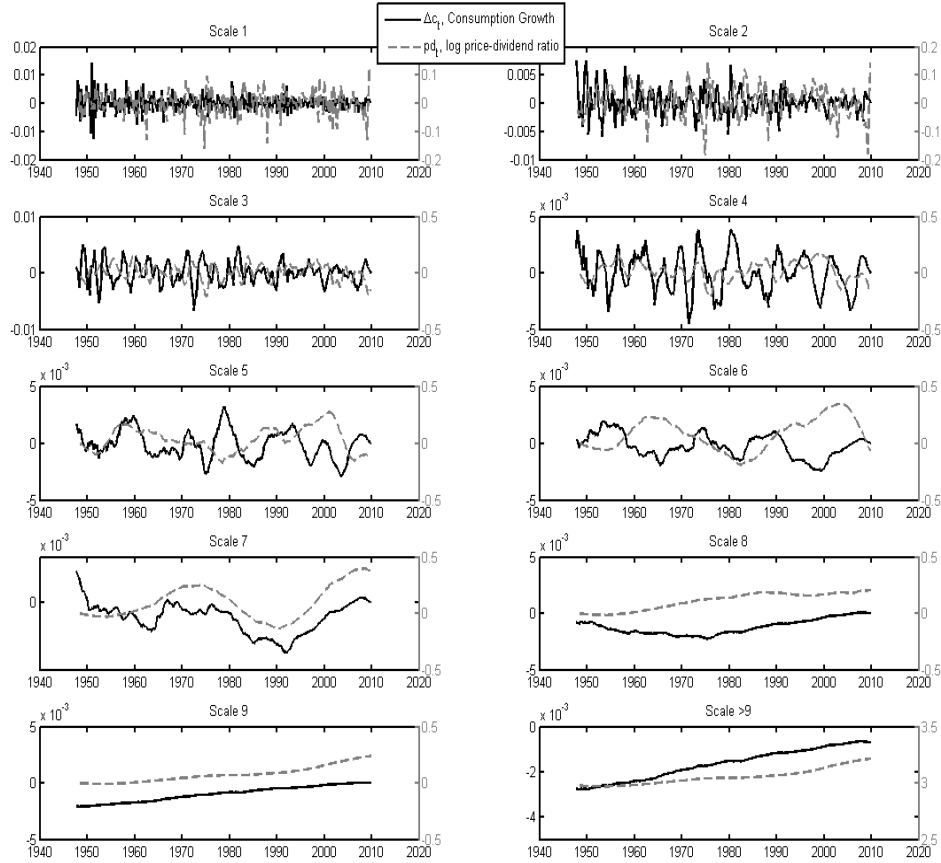


Figure 3: Time-scale decomposition for the log price-dividend  $pd_t$  and log consumption growth based upon quarterly data. The figure displays the components of consumption growth  $g_{j,t+2j}$  obtained using the forward decomposition along with the components of the price-dividend ratio  $z_{j,t}^m$  obtained using the backward decomposition. The sample spans the period 1947Q2-2009Q4.

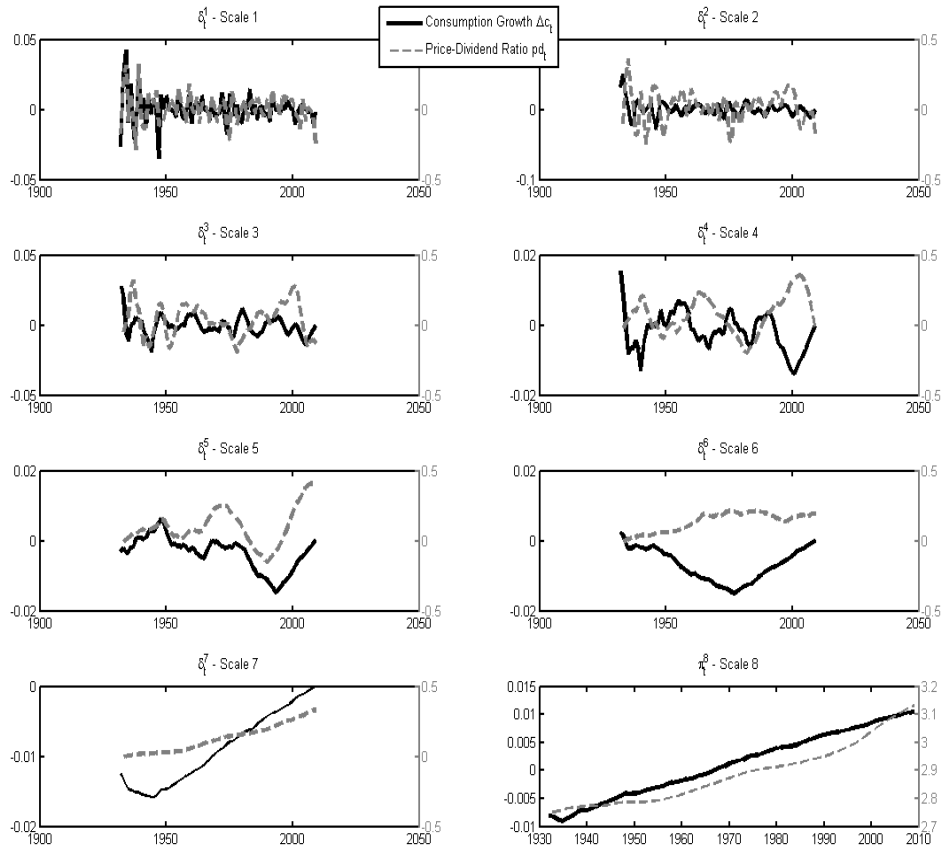


Figure 4: Time-scale decomposition for the log price-dividend  $pd_t$  and log consumption growth  $g_t$  based upon annual data. The figure displays the components of consumption growth  $g_{j,t+2j}$  obtained using the forward decomposition along with the components of the price-dividend ratio  $z_{j,t}^m$  obtained using the backward decomposition. The sample spans the period 1930-2009.

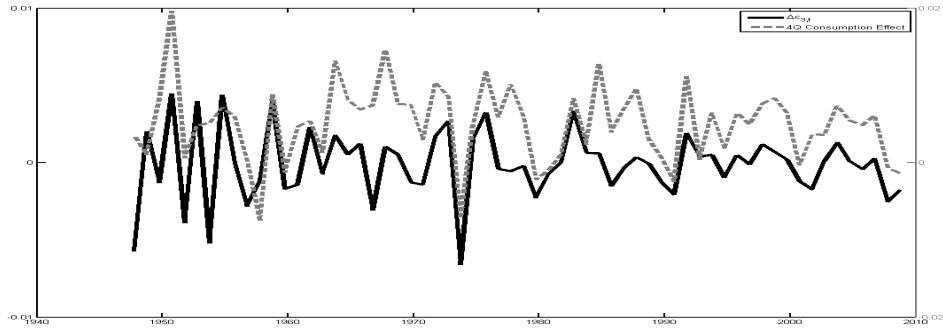


Figure 5: The figure displays the third component of consumption growth,  $g_{3,t}$  along with the real consumption from the third quarter of a calendar year to the fourth quarter as suggested in Rangvid (2010).

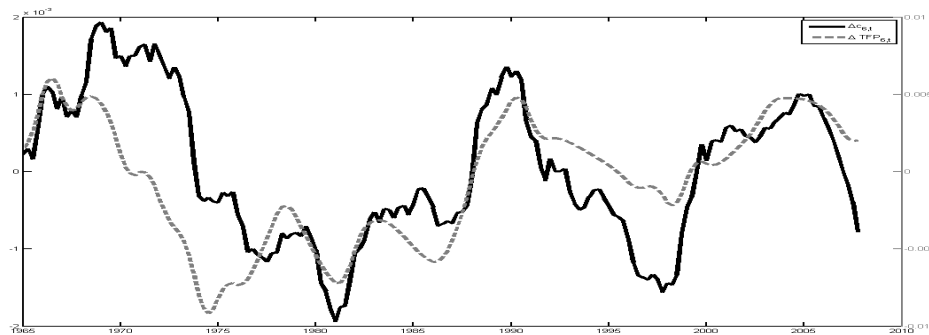


Figure 6: The figure displays the sixth component of consumption growth,  $g_{6,t}$  and total factor productivity,  $\Delta TFP_{6,t}$ .

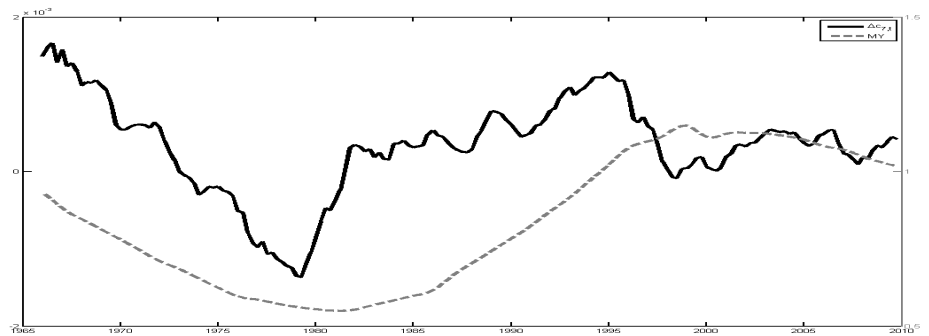


Figure 7: The figure displays the seventh component of consumption growth,  $g_{7,t}$  along with a demographic variable,  $MY_t$ , the middle-aged to young ratio proposed in Geanakoplos et al. (2004).

Periodogram of differences in log consumption

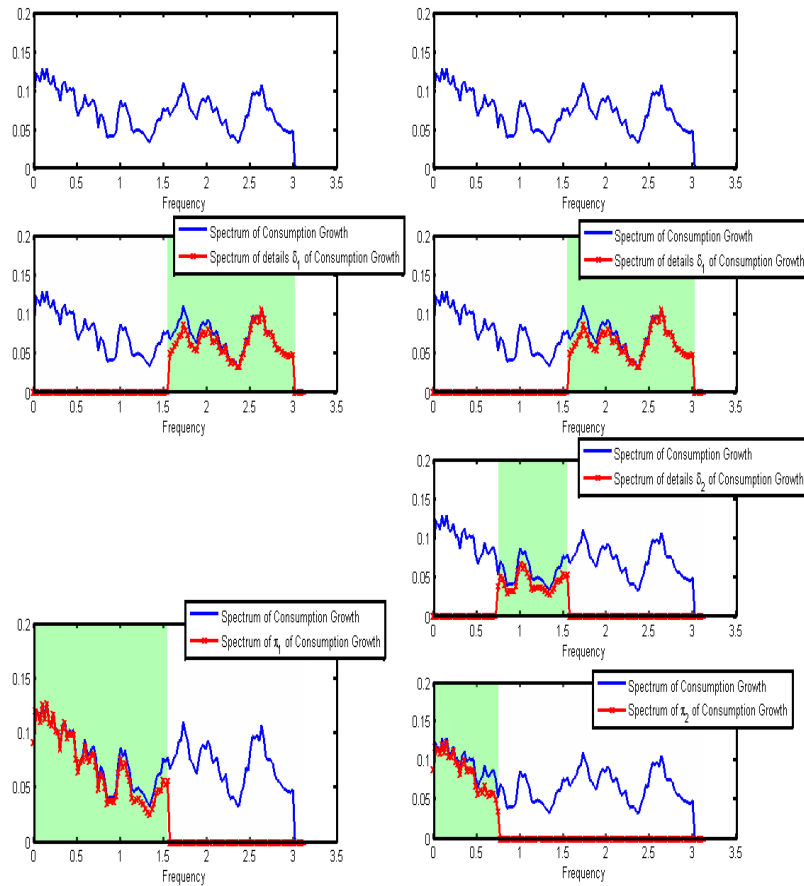


Figure 8: The figure displays the effects of the persistence based decomposition of the consumption time series applied up to level  $J = 1$  (left panels) and  $J = 2$  (right panels). In particular the top panels displays the smoothed periodogram the consumption process for the data. An equally weighted “nearest neighbor” kernel was used to perform the smoothing, equally weighting the 4 nearest frequencies. The bottom right panel displays the Fourier spectrum of the time series  $\pi_t^{(2)}$  whereas the bottom left panel displays the Fourier spectrum of the time series  $\pi_t^{(1)}$ .

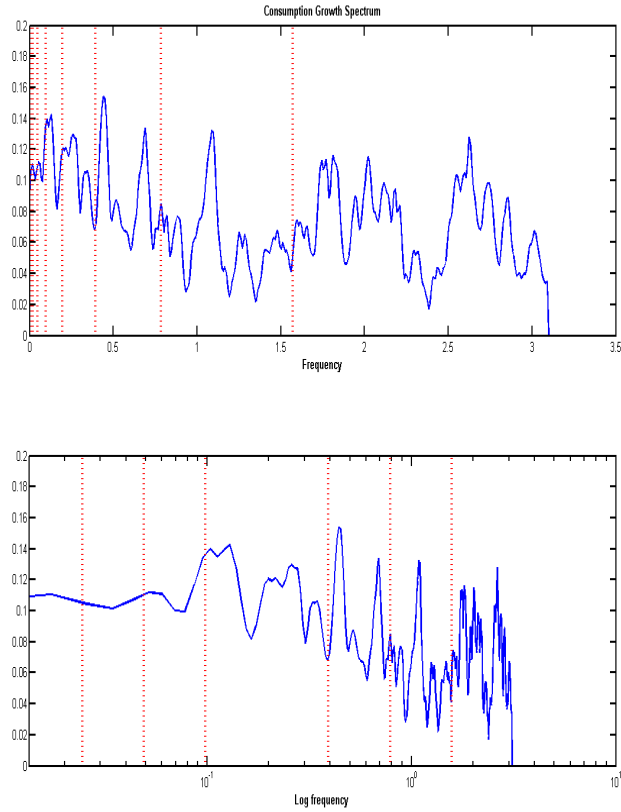


Figure 9: The figure displays the smoothed periodogram the consumption process for the data together with the intervals  $\left[\frac{f_{max}}{2^j}, \frac{f_{max}}{2^{j-1}}\right)$   $j = 1, \dots, 8$ . An equally weighted “nearest neighbor” kernel was used to perform the smoothing, equally weighting the 4 nearest frequencies. In the top panel linear scale is used for frequencies whereas in the bottom panel logarithmic scale is used for the X-axis.

Component	Quarterly-frequency resolution
$x_{1,t}$	1 – 2 quarters
$x_{2,t}$	2 – 4 quarters
$x_{3,t}$	1 – 2 years
$x_{4,t}$	2 – 4 years
$x_{5,t}$	4 – 8 years
$x_{6,t}$	8 – 16 years
$x_{7,t}$	16 – 32 years
$x_{8,t}$	32 – 64 years
$\pi_t^{(8)}$	> 64 years

Table 1: Frequency interpretation of the component  $x_{j,t}$  at level of persistence  $j$ . We assume the original time series  $x_t$  to be observed at quarterly intervals.

$j =$	Persistence level $j$							
	1	2	3	4	5	6	7	8
$g_{1,t+1}$	-	0.0495 (0.435)	-0.0040 (0.949)	-0.0100 (0.875)	-0.0257 (0.685)	-0.0194 (0.760)	-0.0221 (0.728)	-0.0129 (0.839)
$g_{2,t+1}$	-	-	<b>0.1310</b> (0.0380)	-0.0372 (0.558)	-0.0632 (0.319)	-0.0430 (0.498)	-0.0413 (0.515)	-0.0373 (0.557)
$g_{3,t+1}$	-	-	-	<b>0.1320</b> (0.0366)	-0.0646 (0.308)	0.0074 (0.907)	-0.0841 (0.184)	-0.0643 (0.310)
$g_{4,t+1}$	-	-	-	-	0.0801 (0.206)	0.0523 (0.409)	-0.0906 (0.152)	-0.1223 (0.067)
$g_{5,t+1}$	-	-	-	-	-	0.1016 (0.108)	-0.0699 (0.269)	-0.2025 (0.110)
$g_{6,t+1}$	-	-	-	-	-	-	0.0483 (0.446)	-0.2757 (0.121)
$g_{7,t+1}$	-	-	-	-	-	-	-	-0.3018 (0.089)

Table 2: This table reports the Pearson’s correlation coefficients among consumption growth components.

$j =$	Component at persistence level $j$							
	1	2	3	4	5	6	7	8
$\frac{Var(g_{j,t})}{Var(\sum g_{j,t})}$	0.314	0.183	0.147	0.112	0.066	0.050	0.052	0.075
$\frac{Var(z_{j,t}^m)}{Var(\sum z_{j,t}^m)}$	0.017	0.028	0.045	0.065	0.120	0.249	0.305	0.171

Table 3: Contribution to total unconditional variance of the different details components  $g_{j,t}$  of the log consumption growth. Note that  $Var(\sum g_{j,t}) = Var(g_t)$  and  $Var(\sum z_{j,t}^m) = Var(z_t^m)$

$z_t =$	$\rho_j^g$	$\bar{R}^2$
$g_{1,t+1}$	-0.454 (-5.40)	[0.20]
$g_{2,t+1}$	0.350 (8.12)	[0.12]
$g_{3,t+1}$	0.746 (14.93)	[0.55]
$g_{4,t+1}$	0.885 (28.12)	[0.81]
$g_{5,t+1}$	0.966 (44.91)	[0.93]
$g_{6,t+1}$	0.985 (65.20)	[0.97]
$g_{7,t+1}$	0.992 (94.06)	[0.99]
$g_{8,t+1}$	0.998 (130.05)	[0.99]

Table 4: This table reports the results of regressions of each of the components of 1-period ahead consumption growth  $g_{j,t+1}$  on its own lagged components  $g_{j,t}$ . For each regression, the table reports OLS estimates of the regressors, Hansen and Hodrick corrected t-statistics in parentheses and adjusted  $R^2$  statistics in square brackets. The effective sample is quarterly and spans the period 1947Q2-2007Q4.

$z_t =$	$\rho_j^{pd}$	$\bar{R}^2$
$pd_{1,t+1}$	0.286 (1.92)	[0.10]
$pd_{2,t+1}$	0.763 (12.45)	[0.43]
$pd_{3,t+1}$	0.892 (20.01)	[0.79]
$pd_{4,t+1}$	0.921 (31.51)	[0.93]
$pd_{5,t+1}$	0.954 (68.10)	[0.99]
$pd_{6,t+1}$	0.981 (64.40)	[0.99]
$pd_{7,t+1}$	0.992 (164.54)	[1.00]
$pd_{8,t+1}$	0.998 (327.63)	[1.00]

Table 5: This table reports the results of regressions of each of the components of 1-period ahead  $pd_{j,t+1}$  on its own lagged components  $pd_{j,t}$ . For each regression, the table reports OLS estimates of the regressors, Hansen and Hodrick corrected t-statistics in parentheses and adjusted  $R^2$  statistics in square brackets. The effective sample is quarterly and spans the period 1947Q2-2009Q4.



$j =$	Persistence level $j$							
	1	2	3	4	5	6	7	8
$pd_t$	0.31	-0.49	-0.73	0.16	-0.17	-0.35	0.28	0.40
	(0.74)	(-1.75)	(-2.88)	(0.50)	(-0.85)	(-1.93)	(2.56)	(1.51)
	[0.00]	[0.01]	[0.06]	[0.01]	[0.02]	[0.24]	[0.38]	[0.01]

Table 6: This table reports the results of predictive regressions of the components of consumption growth  $g_{j,t+2j}$  on the components of (log) price-dividend ratio  $pd_{j,t}$ . For each regression, the table reports OLS estimates of the regressors, Hansen and Hodrick corrected t-statistics in parentheses and adjusted  $R^2$  statistics in square brackets. The effective sample is quarterly and spans the period 1947Q2-2009Q4.

$j =$	Persistence level $j$							
	1	2	3	4	5	6	7	8
$pc_t$	0.40	-0.25	-0.67	-0.03	-0.11	-0.17	0.27	-0.11
	(1.22)	(-1.11)	(-2.95)	(-0.09)	(-0.77)	(-1.88)	(3.50)	(0.73)
	[0.00]	[0.00]	[0.06]	[0.00]	[0.01]	[0.07]	[0.50]	[0.02]

Table 7: This table reports the results of predictive regressions of the components of 1-period ahead consumption growth  $g_{j,t+2j}$  on the components of (log) price-consumption ratio  $pc_{j,t}$ . For each regression, the table reports OLS estimates of the regressors, Hansen and Hodrick corrected t-statistics in parentheses and adjusted  $R^2$  statistics in square brackets. The effective sample is quarterly and spans the period 1947Q2-2009Q4.

$j =$	Persistence level $j$					
	1	2	3	4	5	6
$pd_t$	5.95	-2.63	1.10	-2.50	1.58	-4.81
	(4.19)	(-3.35)	(1.10)	(-2.69)	(2.90)	(-1.51)
	[0.38]	[0.08]	[0.04]	[0.39]	[0.19]	[0.49]

Table 8: This table reports the results of predictive regressions of the components of consumption growth on the components of (log) price-dividend ratio. For each regression, the table reports OLS estimates of the regressors, Hansen and Hodrick corrected t-statistics in parentheses and adjusted  $R^2$  statistics in square brackets. The sample is annual and spans the period 1930-2009.

$j =$	Persistence level $j$							
	1	2	3	4	5	6	7	8
$pd_t$	0.39	-0.58	-0.78	0.17	-0.17	-0.37	0.29	0.31
	(0.86)	(-1.91)	(-2.96)	(0.51)	(-0.87)	(-1.98)	(2.55)	(1.17)
	[0.00]	[0.02]	[0.06]	[0.01]	[0.02]	[0.25]	[0.38]	[0.05]

Table 9: This table reports the results of predictive regressions of the components of dividend growth  $gd_{j,t+2j}$  on the components of (log) price-dividend ratio  $pd_{j,t}$ . For each regression, the table reports OLS estimates of the regressors, Hansen and Hodrick corrected t-statistics in parentheses and adjusted  $R^2$  statistics in square brackets. The effective sample is quarterly and spans the period 1947Q2-2009Q4.

$j =$	Persistence level $j$							
	1	2	3	4	5	6	7	8
$pd_t$	0.93	-1.63	-1.58	1.78	-1.05	-0.53	1.68	-0.81
	(1.47)	(-2.70)	(-1.78)	(2.55)	(-2.01)	(-2.34)	(19.24)	(-1.06)
	[0.01]	[0.03]	[0.02]	[0.03]	[0.02]	[0.06]	[0.61]	[0.01]

Table 10: This table reports the results of predictive regressions of the components of consumption growth on the components of (log) price-dividend ratio. For each regression, the table reports OLS estimates of the regressors, Hansen and Hodrick corrected t-statistics in parentheses and adjusted  $R^2$  statistics in square brackets. The sample is quarterly and spans the period 1947Q2-2008Q4. The components are extracted using the default filter recommended in Christiano and Fitzgerald (1999).

Asset	Sample	$r_{f,j,t+1} = \alpha_f + \frac{1}{\psi} g_{t+1,j}$ $\hat{\psi}$
$r_{f,t+1}$	1947Q2-2009Q4	4.762 (6.505)
$r_{f,t+1}$	1978Q1-2009Q4	4.594 (3.037)
$r_{f,t+1}$	1947Q2-2009Q4	5.076 (2.707)

Table 11: This table displays the EIS estimates using the risk free rate. The first row reports two-stage least squares estimates. The instruments are the components of consumption growth and of the returns for the asset. The second and third rows report the estimate based on not-redundant persistence-based decomposition as suggested in Fadili and Bullmore (2002) based on 128 and 256 data points respectively.

Scale $j =$	Half-life (Years)	$Q_j$ (1.0e-005)	Risk Exposure (1.0e-003)	Risk Price	Risk Premium (%)
1	0.08	0.31	1.072	4.67	0.50
2	0.44	0.18	0.712	12.12	0.86
3	1.52	0.15	0.592	32.33	1.91
4	3.63	0.12	0.652	96.03	6.29
5	4.57	0.07	0.288	168.69	4.86
6	12.5	0.05	0.140	181.71	2.51
7	18.77	0.05	0.068	183.28	1.25
8	33.27	0.07	0.016	183.84	0.26

Table 12: This table reports equity premium (in %)  $E_t[r_{m,t+1} - r_{f,t}]$  decomposed by level of persistence. We set  $\psi = 1.5$  and  $\gamma = 5$ .

## A The Persistence Based Decomposition

### A. The Backward and Forward Decomposition

In this section we show that the identities in equations (6) and (7) hold true. It is useful to carry out the calculations in the simple case where  $J = 2$  which is enough to get the point and does not mess up the algebra. Generalization are straightforward and can be found in Daubechies (1992), Daubechies (1990), Mallat (1989a) and Mallat (1989b).

When  $J = 2$  the identity (6) becomes

$$x_t = x_{2,t} + x_{1,t} + \pi_t^{(2)}$$

To construct the components  $x_{j,t}$  and  $\pi_t^{(2)}$  we use equations (4) and (5). Using these equations one obtains

$$\begin{aligned}\pi_t^{(2)} &= \frac{x_t + x_{t-1} + x_{t-2} + x_{t-3}}{4} \\ x_{2,t} &= \frac{x_t + x_{t-1} - x_{t-2} - x_{t-3}}{4} \\ x_{1,t} &= \frac{x_t - x_{t-1}}{2}\end{aligned}$$

It is trivial to verify that (6) holds. In order to show that relation (7) holds it is useful to introduce a linear transformation which maps a block of  $2^J$  observations, i.e.  $\mathbf{x} = \{x_{t-i}\}_{i=0}^{2^J-1}$  into its components  $\{x_{j,t}\}_{j=1}^J$  with persistence  $j$  and the permanent component  $\pi_t^{(J)}$ .<sup>38</sup> This linear mapping can be explicitly represented using a matrix of size  $2^J$ ,  $\mathcal{T}^{(J)}$ . While we refer once again to Daubechies (1992), Daubechies (1990), Mallat (1989a) and Mallat (1989b) for the construction of  $\mathcal{T}^{(J)}$  in the general case, to clarify our approach we exemplify its construction for  $J = 2$ . In this case the matrix  $\mathcal{T}^{(2)}$  that maps the time series in the components with different levels of persistence  $j$  is given by:

$$\mathcal{T}^{(2)} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (\text{A.1})$$

In order to verify that the above matrix is the transformation we are after, we first let  $\underline{X}_t^{(2)}$  be the vector that collects the elements of our time series from time  $t - 2^2 + 1$  to time  $t$

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<sup>38</sup>Indeed the linear transformation maps the original windowed time series with  $2^J$  elements into exactly  $2^J$  components. This is because at level of persistence  $j$  we will have exactly  $2^J/2^j$  independent fluctuations.

(hence backward from time  $t$ ), i.e.

$$\underline{X}_t^{(2)} = \begin{bmatrix} x_{t-3} \\ x_{t-2} \\ x_{t-1} \\ x_t \end{bmatrix}$$

and then we postmultiply it by the matrix  $\mathcal{T}^{(2)}$

$$\underline{\tilde{X}}_t^{(2)} = \mathcal{T}^{(2)} \underline{X}_t^{(2)}$$

to obtain

$$\underline{\tilde{X}}_t^{(2)} = \begin{pmatrix} \pi_t^{(2)} \\ x_{2,t} \\ x_{1,t-2} \\ x_{1,t} \end{pmatrix} = \begin{pmatrix} \frac{x_t + x_{t-1} + x_{t-2} + x_{t-3}}{4} \\ \frac{x_t + x_{t-1} - x_{t-2} - x_{t-3}}{4} \\ \frac{x_{t-2} - x_{t-3}}{2} \\ \frac{x_t - x_{t-1}}{2} \end{pmatrix}$$

We have therefore verified that the vector  $\underline{\tilde{X}}_t^{(2)}$  that stacks the sequence of components  $x_{2,t}, x_{1,t-2}, x_{1,t}$  below the persistent component  $\{\pi_t^{(2)}\}$  can be obtained as a linear transformation of our original time series. Importantly we remark that our transformation matrix  $\mathcal{T}^{(2)}$  does not depend on the time  $t$  from which we start collecting (backward) the time series observations.

Next we show that the forward persistence decomposition in equation (7) which is reported here for reader convenience holds:

$$x_{t+1} = \sum_{j=1}^J x_{j,t+2j} + \pi_{t+2^J}^{(J)}$$

Once again we give a simple example for the case  $J = 2$ . In order to obtain the components  $x_{1,t+2}, x_{2,t+4}, \pi_{t+4}^{(2)}$  we can still use the transformation matrix given by equation (A.1) but now we apply it to the vector that collects the elements of our time series from time  $t + 1$  to time  $t + 2^J$  (hence forward from time  $t + 1$ ), i.e.

$$\underline{X}_{t+1}^{(2)} = \begin{bmatrix} x_{t+4} \\ x_{t+3} \\ x_{t+2} \\ x_{t+1} \end{bmatrix}$$

By doing so we obtain

$$\begin{aligned}\pi_{t+4}^{(2)} &= \frac{x_{t+4} + x_{t+3} + x_{t+2} + x_{t+1}}{4} \\ x_{2,t+4} &= \frac{-x_{t+4} - x_{t+3} + x_{t+2} + x_{t+1}}{4} \\ x_{1,t+2} &= \frac{-x_{t+2} + x_{t+1}}{2} \\ x_{1,t+4} &= \frac{-x_{t+4} + x_{t+3}}{2}\end{aligned}$$

Finally we can check that for  $J = 2$

$$\begin{aligned}x_{t+1} &= \sum_{j=1}^J x_{j,t+2j} + \pi_{t+2J}^{(J)} \\ &= x_{2,t+4} + x_{1,t+2} + \pi_{t+4}^{(2)}\end{aligned}$$

holds true.

### B. A Frequency Interpretation of the Persistence-based Decomposition

The filtering procedure described in Section **II.B.1.** and the persistence properties of the series  $\pi_t^{(j)}$  and  $x_{j,t}$  can be usefully visualized in the frequency domain in terms of their Fourier spectra.<sup>39</sup> As an example we apply equations (4) and (5) for the case  $J = 1$  and  $J = 2$  to the consumption growth time series and we report the results in the left and right columns of Figure 8 respectively.

The top subplot of Figure 8 shows the Fourier spectrum of the aggregate consumption growth time series. The shadowed region in the bottom left panel identifies the part of the spectrum which survives after the first application of the moving average filter, namely the spectrum of  $\boldsymbol{\pi}_t^{(1)} = \{\pi_t^{(1)}\}_{t \in \mathbb{Z}}$ . We clearly see that the effect of the simple 2-period moving average is to halve the spectrum and to keep the lowest part. Nevertheless the high frequency part of the spectrum is recovered by the component  $\mathbf{x}_{1,t} = \{x_{1,t}\}_{t \in \mathbb{Z}}$ . This can be seen in the mid left panel where the shadowed region represent the spectrum of  $\mathbf{x}_{1,t}$ . In some sense we are reassured that, for  $J = 1$  we recover a simple permanent-transitory decomposition.

[Insert Figure 8 about here.]

The right column of Figure 8 shows the case  $J = 2$  where we filter out the first two compo-

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<sup>39</sup>Indeed in the frequency representation, a  $2^j$ -period moving average operator works as a low band pass filter which removes all those components whose frequency is higher than  $2^j$ .

nents  $\mathbf{x}_{1,t}$  and  $\mathbf{x}_{2,t}$  of the the aggregate consumption growth. From (5) we know that  $\mathbf{x}_{2,t}$  is obtained as the difference of  $\boldsymbol{\pi}_t^{(2)}$  and  $\boldsymbol{\pi}_t^{(1)}$  reported in the bottom right and left panels, respectively. The third left panel displays exactly this operation and confirms our previous intuition that by taking the difference between  $\boldsymbol{\pi}_t^{(2)}$  and  $\boldsymbol{\pi}_t^{(1)}$ , we are able to identify the fluctuations of the original time series  $\mathbf{x}$  that lies in the well defined frequency range  $[1/4, 1/2]$ .<sup>40</sup> The bottom panels of Figure 8 show that the Fourier spectrum of the time series  $\boldsymbol{\pi}_t^{(2)}$  differs from that of  $\boldsymbol{\pi}_t^{(1)}$  because the dyadic averaging operation gets rid of the components at frequencies larger than  $1/2^2$  while the low frequency components are essentially left unaffected. Therefore we conclude that increasing the window of values over which the average is made is equivalent to focus one's attention on lower and lower frequencies.

## B The Long-run Risk Model with Persistence Heterogeneity

In this Section we show the steps to obtain the values of the financial ratios coefficients  $A_{0,j}, A_j, A_{0,j}^m, A_j^m$  in terms of the parameters of the model. We then compute the equity premia on both the consumption claim asset and the market return. Finally we derive the risk-free rate.

### A. The Financial Ratios

We solve first for the price-consumption coefficients  $A_{0,j}, A_j$  and then for the price-dividend ones  $A_{0,j}^m, A_j^m$ . To obtain the values of the coefficients  $A_{0,j}, A_j$  we exploit the Euler condition

$$E_t \left[ \exp \left( \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + \theta r_{a,t+1} \right) \right] = 1$$

which is derived from (11) for the special case where the asset being priced is the aggregate consumption claim, i.e.  $r_{i,t+1} = r_{a,t+1}$ . We then express the log consumption growth  $g_{t+1}$  and the return  $r_{a,t+1}$  in terms of the factors  $\{x_{j,t}\}_j$  and of the innovations  $\{e_{j,t+2j}^g\}_j$  and  $\{\varepsilon_{j,t+2j}\}_j$ . To do so we plug first the Campbell and Shiller (1988) approximation for log returns, see equation (13), into the above expression to obtain:

$$E_t \left[ \exp \left( \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + \theta \underbrace{(\kappa_0 + \kappa_1 z_{t+1}^a - z_t^a + g_{t+1})}_{r_{a,t+1}} \right) \right] = 1$$

---

<sup>40</sup>Each component has in fact a corresponding Fourier spectrum localized in the finite interval of frequencies  $\left[ \frac{f_{max}}{2^j}, \frac{f_{max}}{2^{j-1}} \right)$  where  $f_{max}$  is the maximum frequency of observations, and in our case quarterly.

By the backward decomposition (6) applied to the (demeaned) price-consumption ratio at time  $t$  and by the forward decomposition (7) applied to the (demeaned) consumption growth and price-consumption processes at time  $t + 1$  we have:

$$z_t^a = \sum_{j=1}^J z_{j,t}^a \quad (\text{B.1})$$

$$z_{t+1}^a = \sum_{j=1}^J z_{j,t+2j}^a \quad (\text{B.2})$$

$$g_{t+1} = \sum_{j=1}^J g_{j,t+2j} \quad (\text{B.3})$$

Plugging the above expressions into the Euler condition yields:

$$E_t \left[ \exp \left( \theta \log \beta - \frac{\theta}{\psi} \left( \sum_{j=1}^J g_{j,t+2j} \right) + \theta \left( \kappa_0 + \kappa_1 \left( \sum_{j=1}^J z_{j,t+2j} \right) - \left( \sum_{j=1}^J z_{j,t} \right) + \left( \sum_{j=1}^J g_{j,t+2j} \right) \right) \right) \right] = 1$$

Finally using the dynamics for the components of log consumption growth given in equation (8) together with our guess for the components of price-consumption ratio solution given in equation (14), rearranging terms and using the log normal properties of the shocks we obtain:

$$\begin{aligned} & E_t \left[ \exp \left( \theta \log \beta + \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J g_{j,t+2j} \right) + \right. \right. \\ & \quad \left. \left. \theta \left( \kappa_0 + \kappa_1 \left( \sum_{j=1}^J z_{j,t+2j} \right) - \left( \sum_{j=1}^J z_{j,t} \right) \right) \right) \right] \\ &= E_t \left[ \exp \left( \theta \log \beta + \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J x_{j,t} + \sum_{j=1}^J e_{j,t+2j}^g \right) + \right. \right. \\ & \quad \left. \left. \theta \left( \kappa_0 + \kappa_1 \left( \sum_{j=1}^J A_{0,j} + \sum_{j=1}^J A_j x_{j,t+2j} \right) - \left( \sum_{j=1}^J A_{0,j} + \sum_{j=1}^J A_j x_{j,t} \right) \right) \right) \right] \end{aligned}$$



$$\begin{aligned}
&= E_t \left[ \exp \left( \theta(\log \beta + \kappa_0 + (\kappa_1 - 1) \sum_{j=1}^J A_{0,j}) + \dots \right. \right. \\
&\quad \left. \left. \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J x_{j,t} + \sum_{j=1}^J e_{j,t+2j}^g \right) + \theta \left( \kappa_1 \sum_{j=1}^J A_j x_{j,t+2j} - \sum_{j=1}^J A_j x_{j,t} \right) \right) \right] \\
&= E_t \left[ \exp \left( \theta(\log \beta + \kappa_0 + (\kappa_1 - 1) \sum_{j=1}^J A_{0,j}) + \dots \right. \right. \\
&\quad \left. \left. \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J x_{j,t} + \sum_{j=1}^J e_{j,t+2j}^g \right) + \theta \left( \kappa_1 \sum_{j=1}^J A_j \underbrace{(e_j M \tilde{X}_t + e_j \varepsilon_{t+2j})}_{x_{j,t+2j}} - \sum_{j=1}^J A_j x_{j,t} \right) \right) \right] = 1
\end{aligned}$$

where we defined  $\tilde{X}_t \equiv [x_{1,t}, \dots, x_{J,t}]^\top$ . Collecting terms in  $\tilde{X}_t$  yields eventually a system of equations

$$e_j \left( \left( 1 - \frac{1}{\psi} \right) + A_j (\kappa_1 M - \mathbb{I}_J) \right) = 0$$

for all  $j = 1, \dots, J$ . If we introduce the following column vectors

$$\underline{A} \equiv [A_1, \dots, A_J]^\top$$

the solution to these equations is given by the following vectors of sensitivities:

$$\underline{A} = \left( 1 - \frac{1}{\psi} \right) \underline{1} (\mathbb{I}_J - \kappa_1 M)^{-1}$$

To derive the expression for  $A_j^m$  we exploit once again the Euler condition

$$E_t \left[ \exp \left( \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} + r_{m,t+1} \right) \right] = 1$$

where now the asset being priced is the market return  $r_{m,t+1}$ . Following the same steps as above, and additionally using the Campbell and Shiller (1988) log-linear approximation for

$r_{m,t+1}$ , see equation (13) we can then rewrite the Euler equation as:

$$E_t \left[ \exp \left( \theta \log \beta + \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J g_{j,t+2j} \right) + \right. \right. \\ \left. \left. (\theta - 1) \left( \kappa_0 + \kappa_1 \left( \sum_{j=1}^J z_{j,t+2j} \right) - \left( \sum_{j=1}^J z_{j,t} \right) \right) - \left( \sum_{j=1}^J g_{j,t+2j} \right) + \right. \right. \\ \left. \left. \underbrace{\kappa_{0,m} + \kappa_{1,m} \left( \sum_{j=1}^J z_{j,t+2j}^m \right) - \left( \sum_{j=1}^J z_{j,t}^m \right) + \left( \sum_{j=1}^J g_{j,t+2j} \right)}_{r_{m,t+1}} \right) \right] =$$

Let's focus on the term

$$\theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J g_{j,t+2j} \right) + (\theta - 1) \left( \kappa_0 + \kappa_1 \left( \sum_{j=1}^J z_{j,t+2j} \right) - \left( \sum_{j=1}^J z_{j,t} \right) \right)$$

This can be written (neglecting error terms that are going to be captured by the constant using the law of log normal distribution and neglecting constant terms) as follows

$$= \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J x_{j,t} \right) + (\theta - 1) \left( \kappa_1 \left( \sum_{j=1}^J A_j x_{j,t+2j} \right) - \left( \sum_{j=1}^J A_j x_{j,t} \right) \right) \\ = \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J x_{j,t} \right) + (\theta - 1) \left( \underline{A} (\kappa_1 M - \mathbb{I}_J) \tilde{X}_t \right)$$

and plugging the solution for  $\underline{A}$  we eventually obtain

$$= \theta \left( 1 - \frac{1}{\psi} \right) \left( \sum_{j=1}^J x_{j,t} \right) - (\theta - 1) \left( 1 - \frac{1}{\psi} \right) \underline{1} \tilde{X}_t \\ = \left( 1 - \frac{1}{\psi} \right) \underline{1} \tilde{X}_t$$

Plugging into the Euler equation the above simplifying expression, using the dynamics for the components of the log consumption growth given in formula (8) and rearranging terms

we have:

$$\begin{aligned}
& E_t \left[ \exp \left( \theta \log \beta + \left(1 - \frac{1}{\psi}\right) \underline{1}\tilde{X}_t - \left(\sum_{j=1}^J g_{j,t+2j}\right) \right. \right. \\
& \quad \left. \left. + \kappa_{0,m} + \kappa_{1,m} \left(\sum_{j=1}^J z_{j,t+2j}^m\right) - \left(\sum_{j=1}^J z_{j,t}^m\right) + \left(\sum_{j=1}^J g d_{j,t+2j}\right) \right) \right] \\
&= E_t \left[ \exp \left( \theta \log \beta + \left(1 - \frac{1}{\psi}\right) \underline{1}\tilde{X}_t - \left(\sum_{j=1}^J x_{j,t} + \sum_{j=1}^J e_{j,t+2j}^g\right) \right. \right. \\
& \quad \left. \left. + \kappa_{0,m} + \kappa_{1,m} \left(\sum_{j=1}^J z_{j,t+2j}^m\right) - \left(\sum_{j=1}^J z_{j,t}^m\right) + \left(\sum_{j=1}^J g d_{j,t+2j}\right) \right) \right] \\
&= E_t \left[ \exp \left( \theta \log \beta - \left(\frac{1}{\psi}\right) \underline{1}\tilde{X}_t - \sum_{j=1}^J e_{j,t+2j}^g \right) \right. \\
& \quad \left. + \kappa_{0,m} + \kappa_{1,m} \left(\sum_{j=1}^J z_{j,t+2j}^m\right) - \left(\sum_{j=1}^J z_{j,t}^m\right) + \left(\sum_{j=1}^J g d_{j,t+2j}\right) \right) \right]
\end{aligned}$$

Finally, analogously to what we have done for the return on the consumption claim, using the dynamics for the components of the log dividend growth given in equation (9) together with our guess for the components of log price-dividend ratio given in equation (14), rearranging terms and using the log normal properties of the shocks we obtain:

$$\begin{aligned}
& E_t \left[ \exp \left( \theta \log \beta + \kappa_{0,m} + (\kappa_{1,m} - 1) \sum_{j=1}^J A_{0,j}^m - \left(\frac{1}{\psi}\right) \underline{1}\tilde{X}_t - \sum_{j=1}^J e_{j,t+2j}^g \right) + \right. \\
& \quad \left. \left( \kappa_{1,m} \left(\sum_{j=1}^J A_j x_{j,t+2j}\right) - \left(\sum_{j=1}^J A_j^m x_{j,t}\right) \right) + \left(\sum_{j=1}^J \phi_j x_{j,t} + \sum_{j=1}^J e_{j,t+2j}^d\right) \right) \right] \\
&= E_t \left[ \exp \left( \theta \log \beta + \kappa_{0,m} + (\kappa_{1,m} - 1) A_{0,-1}^m + (\kappa_{1,m} - 1) \sum_{j=1}^J A_{0,j}^m - \left(\frac{1}{\psi}\right) \underline{1}\tilde{X}_t - \sum_{j=1}^J e_{j,t+2j}^g \right) + \right. \\
& \quad \left. \left( \kappa_{1,m} \left(\sum_{j=1}^J A_j^m \underbrace{(e_j M \tilde{X}_t + e_j \epsilon_{t+2j})}_{x_{j,t+2j}}\right) \right) - \left(\sum_{j=1}^J A_j^m x_{j,t}\right) + \left(\sum_{j=1}^J \phi_j x_{j,t} + \sum_{j=1}^J e_{j,t+2j}^d\right) \right) \right] = 1
\end{aligned}$$

Define

$$\begin{aligned}
\underline{A}_m &\equiv [A_1^m, \dots, A_J^m]^\top \\
\underline{\phi} &\equiv [\phi_1, \dots, \phi_J]^\top
\end{aligned} \tag{B.4}$$

In vector notation we have

$$\begin{aligned}\underline{A}^m(\kappa_{1,m}M - \mathbb{I}_J) &= \frac{1}{\psi}\underline{1} - \underline{\phi} \\ \underline{A}^m &= \left(\underline{\phi} - \frac{1}{\psi}\underline{1}\right) (\mathbb{I}_J - \kappa_{1,m}M)^{-1}\end{aligned}$$

### B. The Risk Premia

The risk premium for any asset is determined by the conditional covariance between the return and the SDF. For instance we can compute the risk premium on any asset  $i$  as

$$E_t[r_{i,t+1} - r_{f,t}] + 0.5\sigma_{r_{i,t}}^2 = -cov_t(m_{t+1}, r_{i,t+1})$$

We therefore need to compute first the innovations in the stochastic discount factor and in the returns.

The equilibrium return innovations can be found by plugging the expressions (B.1), (B.2) and (B.3) into the Campbell and Shiller (1988) approximation for log returns, see equation (13) to obtain

$$\begin{aligned}r_{a,t+1} - E_t[r_{a,t+1}] &= \left(\sum_{j=1}^J g_{j,t+2j}\right) + \kappa_0 + \kappa_1 \left(\sum_{j=1}^J z_{j,t+2j}\right) - \left(\sum_{j=1}^J z_{j,t}\right) - E_t[r_{a,t+1}] \\ &= \sum_{j=1}^J e_{j,t+2j}^g + \kappa_1 \left(\sum_{j=1}^J A_j \varepsilon_{j,t+2j}\right) \\ &= e_{t+1}^g + \kappa_1 \underline{A} \boldsymbol{\varepsilon}_{t+1}\end{aligned}\tag{B.5}$$

where in the second line we use our solution for the price-consumption ratio and in the third line we define

$$\begin{aligned}\boldsymbol{\varepsilon}_{t+1}^\top &\equiv [\varepsilon_{1,t+2^1}, \dots, \varepsilon_{J,t+2^J}] \\ e_{t+1}^g &\equiv \sum_j e_{j,t+2j}^g\end{aligned}$$

Analogous steps yield the following expression for the market return innovations

$$r_{m,t+1} - E_t[r_{m,t+1}] = e_{t+1}^d + \kappa_{1,m} \underline{A}_m \cdot \boldsymbol{\varepsilon}_{t+1}\tag{B.6}$$

To find the innovations in the stochastic discount factor, we plug the expressions (B.1), (B.2) and (B.3), together with the dynamics for the components of log consumption growth

given in equation (8) and our guess for the components of price-consumption ratio solution given in equation (14) into equation (12) to obtain:

$$\begin{aligned}
m_{t+1} &= \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} \\
&= \theta \log \beta - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) (\kappa_0 + \kappa_1 z_{t+1}^a - z_t^a + g_{t+1}) \\
&= \theta \log \beta - \frac{\theta}{\psi} \sum_{j=1}^J g_{j,t+2j} + (\theta - 1) \left( \kappa_0 + \kappa_1 \sum_{j=1}^J z_{j,t+2j} - \sum_{j=1}^J z_{j,t} + \sum_{j=1}^J g_{j,t+2j} \right) \\
&= \theta \log(\beta) - \left( 1 - \theta + \frac{\theta}{\psi} \right) \sum_{j=1}^J g_{j,t+2j} + (\theta - 1) \left( \kappa_0 + \kappa_1 \sum_{j=1}^J z_{j,t+2j} - \sum_{j=1}^J z_{j,t} \right) \\
&= \theta \log(\beta) - \left( 1 - \theta + \frac{\theta}{\psi} \right) \sum_{j=1}^J g_{j,t+2j} + (\theta - 1) \left( \kappa_0 + \kappa_1 \sum_{j=1}^J z_{j,t+2j} - \sum_{j=1}^J z_{j,t} \right) \\
&= \theta \log(\beta) - \left( 1 - \theta + \frac{\theta}{\psi} \right) \sum_{j=1}^J g_{j,t+2j} + (\theta - 1) \left( \kappa_0 + \kappa_1 \sum_{j=1}^J A_{0,j} + A_j x_{j,t+2j} - \sum_{j=1}^J A_{0,j} - A_j x_{j,t} \right)
\end{aligned}$$

Finally using the dynamics for our latent factors (10) we obtain

$$\begin{aligned}
m_{t+1} &= \theta \log(\beta) - \left( 1 - \theta + \frac{\theta}{\psi} \right) \sum_{j=1}^J x_{j,t} + (\theta - 1) \left( \kappa_0 + \kappa_1 \sum_{j=1}^J A_{0,j} + A_j \rho_j x_{j,t} - \sum_{j=1}^J A_{0,j} - A_j x_{j,t} \right) \\
&\quad - \left( 1 - \theta + \frac{\theta}{\psi} \right) \sum_{j=1}^J e_{j,t+2j}^g + (\theta - 1) \kappa_1 \left( \sum_{j=1}^J A_j \varepsilon_{j,t+2j} \right)
\end{aligned}$$

which implies

$$\begin{aligned}
m_{t+1} - E_t[m_{t+1}] &= - \left( 1 - \theta + \frac{\theta}{\psi} \right) \sum_{j=1}^J e_{j,t+2j}^g + (\theta - 1) \kappa_1 \left( \sum_{j=1}^J A_j \varepsilon_{j,t+2j} \right) \\
&= -\lambda_\eta \sum_{j=1}^J e_{j,t+2j}^g - \sum_{j=1}^J \lambda_j \varepsilon_{j,t+2j} \\
&= -\lambda_\eta e_{t+1}^g - \underline{\lambda}_n \boldsymbol{\varepsilon}_{t+1}
\end{aligned} \tag{B.7}$$

where

$$\begin{aligned}
\lambda_\eta &\equiv \left( \frac{\theta}{\psi} - \theta + 1 \right) \\
\underline{\lambda}_n &\equiv \kappa_1 (1 - \theta) \underline{A}
\end{aligned}$$

Using the formula for the return on aggregate wealth (B.5) and the innovation in the SDF (B.7) we obtain the risk premium for the consumption claim asset,

$$E_t[r_{a,t+1} - r_{f,t}] + 0.5\sigma_{r_{a,t}}^2 = \lambda_\eta \sigma_{\eta,t}^2 + \kappa_1 \lambda_n \mathbf{Q} \underline{A}'$$

where

$$\begin{aligned}\sigma_\eta^2 &= \text{Var}(e_{t+1}^g) \\ \mathbf{Q} &= \mathbf{E}_t[\boldsymbol{\varepsilon}_{t+1} \boldsymbol{\varepsilon}'_{t+1}]\end{aligned}$$

Similarly to what we have just done, using the formula for the innovations in the market return (B.6) and in the SDF (B.7) the premium to the market return becomes:

$$E_t[r_{m,t+1} - r_{f,t}] + 0.5\sigma_{r_{m,t}}^2 = \kappa_{1,m} \lambda_\varepsilon \mathbf{Q} \underline{A}_m$$

### C. The Risk-Free Rate and The Intertemporal Elasticity of Substitution

To obtain our expression for the risk-free rate we start by plugging the log short-term real interest rate  $r_{f,t+1}$  for  $r_{t+1}^i$  into the Euler equation (11). Then by applying the forward decomposition (7) to the (demeaned) consumption growth and to the log returns processes at time  $t+1$  we observe that the risk-free rate between  $t$  and  $t+1$ ,  $r_{f,t+1}$  satisfies the following condition:

$$E_t \left[ \exp \left( \theta \log \delta - \left( \frac{\theta}{\psi} \right) \sum_{j=1}^J g_{j,t+2j} + (\theta - 1) \sum_{j=1}^J r_{a,j,t+2j} \right) \right] = \exp(-r_{f,t+1})$$

where once again  $r_{a,t+1}$  is the return on the asset that pays consumption as dividend. Taking logs on both sides and using the log normal properties of the shocks we can rewrite it as follows

$$\begin{aligned}r_{f,t+1} &= -\theta \log \beta + \frac{\theta}{\psi} E_t \left[ \sum_{j=1}^J g_{j,t+2j} \right] + (1 - \theta) E_t \left[ \sum_{j=1}^J r_{a,j,t+2j} \right] \\ &\quad - \frac{1}{2} \text{var}_t \left[ \frac{\theta}{\psi} \sum_{j=1}^J g_{j,t+2j} + (1 - \theta) \sum_{j=1}^J r_{a,j,t+2j} \right]\end{aligned}$$

$$\begin{aligned}
&= -\log \beta + \frac{1}{\psi} E_t \left[ \sum_{j=1}^J g_{j,t+2j} \right] + \frac{(1-\theta)}{\theta} E_t \left[ \sum_{j=1}^h r_{a,j,t+2j} - r_f \right] \\
&- \frac{1}{2\theta} \text{var}_t \left[ \frac{\theta}{\psi} \sum_{j=1}^h g_{j,t+2j} + (1-\theta) \sum_{j=1}^h r_{a,j,t+2j} \right]
\end{aligned} \tag{B.8}$$

where in the last line we subtract  $(1-\theta)r_{f,t}$  from both sides and divide by  $\theta$ , where it is assumed that  $\theta \neq 0$ . Further to solve the above expression, note that

$$\text{var}_t \left[ \frac{\theta}{\psi} \sum_{j=1}^J g_{j,t+2j} + (1-\theta) \sum_{j=1}^J r_{a,j,t+2j} \right] = \text{var}_t(m_{t+1})$$

Now we show that in our homoskedastic version of the long-run risk with persistence heterogeneity the variance of the stochastic discount factor  $\text{var}_t(m_{t+1})$  is constant (not function of time). Indeed recall from (B.7) that we have

$$\begin{aligned}
m_{t+1} - E_t[m_{t+1}] &= - \left( \frac{\theta}{\psi} - \theta + 1 \right) (e_{t+1}^g) - \kappa_1(1-\theta)\underline{A} \cdot (\boldsymbol{\varepsilon}_{t+1}) \\
&= -\lambda_\eta(e_{t+1}^g) - \underline{\lambda}_n \cdot (\boldsymbol{\varepsilon}_{t+1})
\end{aligned}$$

from which we can compute the variance as follows

$$\text{var}_t(m_{t+1}) = \lambda_\eta^2 \sigma^2 + \kappa_1^2 (1-\theta)^2 \underline{A} \underline{Q} \underline{A}'$$

Therefore using the dynamics for log consumption growth, namely (8) and reported here for reader's convenience

$$\begin{aligned}
g_{t+1} &= \sum_{j=1}^J g_{j,t+2j} \\
g_{j,t+2j} &= x_{j,t} + e_{j,t+2j}^g \\
e_{j,t+2j}^g &\sim N(0, \sigma_{g,j}^2)
\end{aligned}$$

and taking conditional expectation we rewrite (B.8) as follows:

$$r_{f,t+1} = \alpha_f + \frac{1}{\psi} \sum_j x_{j,t}$$

where  $\alpha_f$  is meant to capture the unconditional mean of the risk-free rate and  $\psi$  is the IES.

## References

- Alvarez, Fernando and Urban J. Jermann**, “Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth,” *Econometrica*, November 2005, *73* (6), 1977–2016.
- Bandi, Federico M. and Benot Perron**, “Long-run risk-return trade-offs,” *Journal of Econometrics*, April 2008, *143* (2), 349–374.
- Bansal, Ravi and Amir Yaron**, “Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles,” *Journal of Finance*, 08 2004, *59* (4), 1481–1509.
- and **Ivan Shaliastovich**, “Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets,” *SSRN eLibrary*, 2010.
- , **Robert Dittmar**, and **Dana Kiku**, “Cointegration and Consumption Risks in Asset Returns,” *Review of Financial Studies*, 2007, *22* (3), 1343–1375.
- , **Robert F. Dittmar**, and **Christian T. Lundblad**, “Consumption, Dividends, and the Cross Section of Equity Returns,” *Journal of Finance*, 08 2005, *60* (4), 1639–1672.
- Beeler, Jason and John Y. Campbell**, “The Long-Run Risks Model and Aggregate Asset Prices: An Empirical Assessment,” NBER Working Papers 14788, National Bureau of Economic Research, Inc March 2009.
- Borovicka, Jaroslav, Lars Peter Hansen, Mark Hendricks, and Jose A. Scheinkman**, “Risk Price Dynamics,” NBER Working Papers 15506, National Bureau of Economic Research, Inc November 2009.
- Calvet, Laurent E. and Adlai J. Fisher**, “Multifrequency news and stock returns,” *Journal of Financial Economics*, October 2007, *86* (1), 178–212.
- Campbell, John Y. and Motohiro Yogo**, “Efficient tests of stock return predictability,” *Journal of Financial Economics*, 2006, *81* (1), 27–60.
- and **N. Gregory Mankiw**, “Consumption, Income, and Interest Rates: Reinterpreting the Time Series Evidence,” NBER Working Papers, National Bureau of Economic Research, Inc May 1990.
- and **Robert J. Shiller**, “Valuation Ratios and the Long-run Stock Market Outlook: An Update,” Cowles Foundation Discussion Papers 1295, Cowles Foundation, Yale University 2001.



- Canova, Fabio**, “Detrending and business cycle facts,” *Journal of Monetary Economics*, May 1998, *41* (3), 475–512.
- Christiano, Lawrence J. and Terry J. Fitzgerald**, “The Band pass filter,” *International Economic Review*, 2003, *44* (2), 435–465.
- Constantinides, George M. and Anisha Ghosh**, “Asset Pricing Tests with Long Run Risks in Consumption Growth,” NBER Working Papers 14543, National Bureau of Economic Research, Inc December 2008.
- Croce, Mariano M.**, “Long-Run Productivity Risk: A New Hope for Production-Based Asset Pricing?,” *SSRN eLibrary*, 2010.
- Daubechies, I.**, “The wavelet transform, time–frequency localization and signal analysis,” *IEEE Trans. Info. Theory*, Sep. 1990, *36* (5), 961–1005.
- Daubechies, Ingrid**, *Ten Lectures on Wavelets (C B M S - N S F Regional Conference Series in Applied Mathematics)*, Soc for Industrial & Applied Math, December 1992.
- Epstein, Larry G and Stanley E Zin**, “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica*, July 1989, *57* (4), 937–69.
- and — , “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis,” *Journal of Political Economy*, April 1991, *99* (2), 263–86.
- Fadili, M.J and E.T Bullmore**, “Wavelet-Generalized Least Squares: A New BLU Estimator of Linear Regression Models with 1/f Errors,” *NeuroImage, Volume 15, Number 1*, 2002.
- Favero, Carlo A., Arie E. Gozluklu, and Andrea Tamoni**, “Long-Run Factors and Fluctuations in Dividend/Price,” *JFQA Forthcoming*, 2010.
- Person, Wayne E., Suresh Nallareddy, and Biqin Xie**, “The Out-of-sample Performance of Long-Run Risk Models,” NBER Working Papers, National Bureau of Economic Research, Inc 2010.
- Garleanu, Nicolae B., Stavros Panageas, and Jianfeng Yu**, “Technological Growth and Asset Pricing,” NBER Working Papers 15340, National Bureau of Economic Research, Inc September 2009.

- Geanakoplos, John, Michael Magill, and Martine Quinzii**, “Demography and the Long-Run Predictability of the Stock Market,” *Brookings Papers on Economic Activity*, 2004, *35* (2004-1), 241–326.
- Gencay, Ramazan and Nikola Gradojevic**, “Errors in Variables Estimation with No Instruments,” Working Paper Series wp30 09, Rimini Centre for Economic Analysis January 2009.
- Hall, Robert E**, “Intertemporal Substitution in Consumption,” *Journal of Political Economy*, April 1988, *96* (2), 339–57.
- Hansen, Lars P. and Robert J. Hodrick**, “Forward Exchange Rates as Optimal Predictors of Future Spot Rates: An Econometric Analysis,” *The Journal of Political Economy*, 1980, *88* (5), 829–853.
- Hansen, Lars Peter and Jose A. Scheinkman**, “Long-term Risk: An Operator Approach,” *Econometrica*, November 2009, *77* (1), 177–234.
- , **John C. Heaton, and Nan Li**, “Consumption Strikes Back? Measuring Long-Run Risk,” *Journal of Political Economy*, 04 2008, *116* (2), 260–302.
- Hsu, Po-Hsuan and Dayong Huang**, “Technology prospects and the cross-section of stock returns,” *Journal of Empirical Finance*, January 2010, *17* (1), 39–53.
- Jagannathan, Ravi and Yong Wang**, “Lazy Investors, Discretionary Consumption, and the Cross-Section of Stock Returns,” *Journal of Finance*, 08 2007, *62* (4), 1623–1661.
- Kaltenbrunner, Georg and Lars A. Lochstoer**, “Long-Run Risk Through Consumption Smoothing,” *Review of Financial Studies*, 2010.
- Koijen, Ralph S.J., Hanno Lustig, and Stijn Van Nieuwerburgh**, “The Cross-Section and Time-Series of Stock and Bond Returns,” Working Paper 15688, National Bureau of Economic Research January 2010.
- , —, —, and **Adrien Verdelhan**, “Long-Run Risk, the Wealth-Consumption Ratio, and the Temporal Pricing of Risk,” *SSRN eLibrary*, 2010.
- Kreps, David M and Evan L Porteus**, “Temporal Resolution of Uncertainty and Dynamic Choice Theory,” *Econometrica*, January 1978, *46* (1), 185–200.
- Kung, Howard and Lukas Schmid**, “Innovation, Growth and Asset Pricing,” 2011.

- Lettau, Martin and Stijn Van Nieuwerburgh**, “Reconciling the Return Predictability Evidence,” *Review of Financial Studies*, July 2008, *21* (4), 1607–1652.
- Lippi, Marco and Paolo Zaffaroni**, “Aggregation of Simple Linear Dynamics: Exact Asymptotic Results,” STICERD - Econometrics Paper Series /1998/350, Suntory and Toyota International Centres for Economics and Related Disciplines, LSE April 1998.
- Lustig, Hanno and Stijn Van-Nieuwerburgh**, “Can Housing Collateral Explain Long-Run Swings in Asset Returns?,” NBER Working Papers 12766, National Bureau of Economic Research, Inc December 2006.
- Mallat, Stephane G.**, “Multiresolution Approximations and Wavelet Orthonormal Bases of  $L_2(\mathbb{R})$ ,” September 1989, *315* (1).
- , “A theory for multiresolution signal decomposition: the wavelet representation,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 1989, *11*, 674–693.
- Moller, Stig V. and Jesper Rangvid**, “The fourth-quarter consumption growth rate,” Technical Report 2010.
- Parker, Jonathan A. and Christian Julliard**, “Consumption Risk and the Cross Section of Expected Returns,” *Journal of Political Economy*, February 2005, *113* (1), 185–222.
- Paseka, Alex and George Theocharides**, “Predictability in Consumption Growth and Equity Returns: A Bayesian Investigation,” *Financial Review*, 2010, *45* (1), 167–203.
- Thornton, Michael**, “Incomplete Information, Long Memory and Aggregation in Consumption under the Permanent Income Hypothesis,” Working Paper 2008.
- Torous, Walter, Rossen Valkanov, and Shu Yan**, “On Predicting Stock Returns with Nearly Integrated Explanatory Variables,” *Journal of Business*, October 2004, *77* (4), 937–966.
- Vissing-Jorgensen, Annette**, “Limited Asset Market Participation and the Elasticity of Intertemporal Substitution,” *Journal of Political Economy*, August 2002, *110* (4), 825–853.