Capital supply uncertainty, cash holdings, and investment*

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Abstract

We develop a model of real investment and cash holdings in which firms face uncertainty regarding their ability to raise funds in the capital markets and have to search for investors when raising outside capital. We provide an explicit characterization of the optimal investment, cash management, and dividend policies for a firm acting in the best interests of incumbent shareholders and show that capital market supply frictions have first-order effects on corporate behavior. We then use the model to explain a key set of stylized facts in corporate finance and to generate a number of novel testable implications relating the supply of funds in capital markets to corporate policy choices.

Keywords: Capital supply uncertainty; cash holdings; real investment; search.

JEL Classification Numbers: D83; G24; G31; G33; G35.

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1 Introduction

Following the seminal contribution of Modigliani and Miller (1958), standard valuation models in corporate finance implicitly assume that capital markets are frictionless so that firms are always able to secure funding for positive net present value (NPV) projects and cash holdings inside the firm are irrelevant. This traditional view has recently been called into question by a large number of empirical studies.[1] These studies document that firms often face uncertainty regarding their future access to capital markets and that this uncertainty has important feedback effects on corporate decisions. They also reveal that the resulting liquidity risk has led firms to accumulate enormous piles of cash over the past decades in response to the increase in idiosyncratic volatility, with an average cash-to-assets ratio for U.S. industrial firms that has increased from 10.5% in 1980 to 23.2% in 2006 (see e.g. Bates, Kahle, and Stulz, 2009).

While it may be clear to most financial economists that capital market frictions can affect corporate policy choices, it is much less clear exactly how they do so. In this paper, we develop a dynamic model of dividend, financing, and investment policies in which the Modigliani and Miller assumption of infinitely elastic supply of capital is relaxed and firms may have to search for investors when raising outside funds. With this model, we seek to understand whether and when capital supply uncertainty affects corporate investment. We are also interested in determining the effects of capital markets frictions on corporate financing and dividend policies, i.e. on the firm's decision to pay out or retain earnings and the firm's decision to issue new securities. By answering these questions, our study aims at understanding whether the supply of capital corresponds to a separate channel through which market imperfections affect corporate behavior.

In order to aid in the intuition of the model, consider the following two settings in which capital supply uncertainty and search frictions are likely to be especially important:

1. **Public equity offerings and capital injections for private firms**: Firms first sell their equity to the public through an initial public offering (IPO). One of the main features of public equity offerings is the book building process, whereby the lead underwriter and senior firm management travel around the country looking for investors. The main objective of this

[1] See for example, Kashyap, Stein and Wilcox (1993), Gan (2007), Becker (2007), Massa, Yasuda, and Zhang (2010), Duchin, Ozbas, and Sensoy (2010), Almeida, Campello, Laranjeira, and Weisbenner (2010), Campello, Graham, and Harvey (2010), Lemmon and Roberts (2010), and Choi, Getmansky, Henderson, and Tookes (2010).

time consuming process is to search for investors until it is unlikely that the issue will fail. Yet, the risk of failure is often not eliminated and a number of IPOs are withdrawn every year. For example, Busaba, Benveniste, and Guo (2001) show that between the mid-1980s and mid-1990s almost one in five IPOs was withdrawn. Evidence from more recent periods suggests that this fraction has increased to over one in two in some years (see e.g. Dunbar and Foerster, 2008). Search frictions are also important for firms that remain private but need new capital injections and must find investors. Indeed, while the initial capital that is required to start a business is usually provided by the entrepreneur and his family, few families have the resources to finance a growing business. When a private company decides to raise private equity capital, it must search for new investors such as angel investors, venture capital firms, or institutional investors. Even when initial outside investors are found, the firm will need to search for additional investors in every subsequent financing round (e.g. second round venture capital), facing here again a significant risk of failure.

2. **Financial crises and economic downturns**: Search frictions are also important for large, publicly traded firms when capital becomes scarce, i.e. during a financial crisis or an economic downturn. The recent global financial crisis has provided a crisp illustration of the potential effects of capital supply (or liquidity) dry ups on corporate behavior, with a number of studies (e.g. Duchin, Ozbas, and Sensoy (2010) or Almeida, Campello, Laranjeira, and Weisbenner (2010)) documenting a significant decline in corporate investment following the onset of the crisis (controlling for firm fixed effects and time-varying measures of investment opportunities). A survey of 1,050 CFOs by Campello, Graham, and Harvey (2010) also indicates that the contraction in capital supply during the recent financial crisis led firms to burn through more cash to fund their operations and to bypass attractive investment opportunities.

Our analysis starts with the observation that when capital supply is uncertain, real investment and survival may depend on a firm's cash holdings. As a result, firms will choose their dividend and retention policies so as to match the future financing needs associated with these two motives, anticipating future financing constraints. To illustrate the implications of this observation for corporate policy choices, we consider a firm with assets in place that generate a continuous stream of stochastic cash flows and a real option to expand operations. The firm is financed with common equity and has the possibility to exercise its real option at any time. While standard corporate

finance models assume that capital markets are frictionless, we consider instead an environment in which the firm faces uncertainty regarding its ability to raise funds and has to search for investors when raising outside financing. Based on these assumptions, the model yields an explicit characterization of the value-maximizing investment, dividend, and financing policies for a firm acting in the best interests of incumbent shareholders and shows that capital supply uncertainty has first-order effects on corporate decisions.

In the model, the firm maximizes its value by making two interrelated decisions: How much cash to hold and whether to finance investment with internal or external funds. That is, the firm can retain earnings or raise outside funds and can be in one of three equity regimes: positive distributions, zero distributions, or equity issuance. If there are no frictions in the capital markets, then firms can raise instantly as much capital as they want and there is no need to safeguard against future liquidity needs by hoarding cash. This is the traditional assumption of the theoretical literature. With capital supply uncertainty, firms find it optimal to hold cash for two motives. First, cash holdings can be used to fund profitable projects when outside funds are unavailable. Second, cash holdings can be used to cover operating losses and avoid inefficient closure.[2] Holding cash nonetheless is costly because of the lower pecuniary return of liquid assets inside the firm. Firms therefore choose their payout policy to balance the benefits of cash holdings with their costs.

The analysis in the paper allows us to derive the value-maximizing level of cash holdings and to relate it to a number of firm and industry characteristics. We highlight the main empirical implications. Consistent with Opler, Pinkowitz, Stulz, and Williamson (OPSW, 1999), our model predicts that cash holdings should increase with cash flow volatility since an increase in volatility leads to an increase in the risk of inefficient closure. The model also predicts that firms with more tangible assets (i.e. firms with a higher liquidation value) should have lower cash holdings and a greater propensity to invest out of internal funds. This prediction results from the fact that an increase in the liquidation value of assets leads to a drop in the cost of inefficient closure. Another interesting prediction of the model is that firms should always increase their cash buffer when raising funds from outside investors. This prediction is consistent with the evidence in Kim and Weisbach (2008) and McLean (2010), who find that firms' decisions to issue equity are essentially driven by their desire to build up cash reserves.

[2]While our basic model considers that it is costless to issue securities, we introduce issuance costs in Section 5, thereby providing an additional motive for holding cash inside the firm.

Another novel prediction of our model is that cash holdings should be used to cover operating losses rather than to finance new investment, consistent with the large sample studies of OPSW (1999) and Bates, Khale and Stulz (BKS, 2009) and with the survey of Lins, Servaes, and Tufano (2010). A direct implication of this result is that cash holdings represent essentially a risk management tool aimed at insuring the firm against potential losses. In fact, we also find that when there is a large benefit to investment (when the NPV of the growth option is large) or when the opportunity cost of not investing is large (i.e. when cash flow volatility is high), it is optimal for firms to accelerate investment with internal funds.

The model also generates a number of predictions relating capital supply to firms' decisions. For example, we find that firms hold more cash when their access to external capital markets is more limited or when issuance costs of securities are larger, in line with OPSW (1999) and BKS (2009). Another prediction of the model is that negative shocks to the supply of capital should hamper investment even if firms have enough slack to finance investment internally. Finally, we find that financially constrained firm may want to engage in risk increasing strategies, in contrast to the results in Froot, Scharfstein, and Stein (1993).

The present paper relates to several strands of the literature. First, it relates to the real options literature, in which it is generally assumed that firms can instantaneously tap capital markets and finance their capital expenditures by diluting equity at no cost or by issuing debt (see Dixit and Pindyck, 1994, for an early survey and Tserlukevich, 2008, Manso, 2008, Morellec and Schuerhoff, 2010, or Carlson, Fisher and Giammarino, 2010, for recent contributions). In these models, it is never optimal for firms to hold cash (whenever there is a cost of holding cash) and firms may end up raising funds infinitely many times from outside investors to cover temporary losses.

Second, our paper relates to the growing literature on costly external finance (see Décamps, Mariotti, Rochet, and Villeneuve, 2008, or Bolton, Chen, and Wang, 2010, for recent contributions). In this literature, firms generally face constant investment and financing opportunity sets. One direct consequence of this assumption is that, depending on whether the costs of external finance are high or low, firms either never raise external funds or are never liquidated. In addition, when the cost of external finance is low, it is optimal to raise external funds only when the firm's cash buffer is depleted. That is, firms never simultaneously hold cash and raise external funds and they only tap external capital market following a series of negative shocks. One key difference between

our analysis and prior contributions is that we propose and solve a dynamic model in which firms find it optimal not only to have cash holdings but also to raise funds (in discrete amounts) from outside capital markets on a regular basis, consistent with the evidence in Fama and French (2005). We then use this model to generate a rich set of testable predictions about firms' cash holdings, financing, and investment policies and the use of cash holdings in corporations.

Finally, our model is related to the study of Décamps and Villeneuve (DV, 2007), who examine the optimal dividend policy of a firm that has no access to external funds and owns a growth option to invest. DV show that in such environments it is optimal for the firm to pay out dividends so as to prevent its cash buffer from exceeding an endogenously determined threshold. In their analysis, DV assume that the firm can never raise outside funds and that investment is irreversible in that the liquidation value of assets is zero. As shown in the present paper, these assumptions have important implications for firms' policy choices. Notably, we show that in the limit in which investment becomes costlessly reversible or in which access to outside capital is unrestricted, it is no longer optimal to hold cash. In addition, while the firm may decide to abandon its growth option for low levels of the cash reserves in DV, it is never optimal to do so in our setup.

Before proceeding, it should also be noted that the capital market frictions we model are in some respects similar to the financing constraints studied for example by Almeida, Campello, and Weisbach (2004) and they have some of the same implications. Models of firm behavior based on financing constraints usually predict that agency conflicts between firm insiders and outside investors may prevent firms from raising enough capital to finance positive NPV projects and lead them to hoard cash. One important feature of these models is that they generally focus on a single motive for holding cash, namely the risk of underinvestment. In addition, only demand factors explain variation in the firm's cash holdings in these models, where demand factors are any firm characteristic that raises the net benefit of cash. In our model, firm behavior can only be explained if one takes into account both demand (firm) and supply (market) factors. In addition, cash reserves serve two motives: financing investment and hedging negative cash flow shocks.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 derives the optimal dividend and financing policies for a firm with no growth option when capital supply is uncertain. Section 4 allows the firm to invest in a growth option and derives the value-maximizing financing, investment, and dividend policies in this context. Section 5 extends the

model to consider issuance costs and time-varying capital supply. Section 6 concludes. The proofs are gathered in the Appendix.

2 Model and assumptions

Throughout the paper, agents are risk neutral and discount cash flows at a constant rate ρ. Time is continuous and uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, P; F)$, with the filtration $F = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions.

We consider a firm with assets in place and a growth option. Assets in place generate a continuous stream of cash flows dX_t before investment as long as the firm is in operation. In particular, we consider that the cumulative cash flow process $(X_t)_{t\geq 0}$ at any time t before investment is given by:

$$X_t = \int_0^t (\mu_0 ds + \sigma dB_s),$$

where B is an F-Brownian motion and (μ_0, σ) are constant parameters representing the mean and volatility of the firm cash flows (a similar specification is used in DeMarzo and Sannikov, 2006, or Gryglewicz, 2011). The growth option allows the firm to increase its income stream from dX_t to $dX_t + (\mu_1 - \mu_0)\, dt$, where $\mu_1 - \mu_0 \geq 0$ determines the growth potential of the firm. The cost of investment is constant and denoted by K. The firm has full flexibility in the timing of investment.

Although its assets may be operated forever, the firm can also choose to abandon them. In the model, abandonment occurs either if the firm finds it optimal to liquidate or if its cash buffer reaches zero following a negative shock to cash flows (i.e. if the firm is in distress). We consider that the liquidation value of assets is $\ell_i = \frac{\varphi \mu_i}{\rho}$, where $\varphi \in [0, 1]$ and $1 - \varphi$ represents a haircut related to the partial irreversibility of investment. When $\varphi = 0$, investment is completely irreversible and the liquidation value of assets is zero. By contrast, when $\varphi = 1$, investment is costlessly reversible and there are no market frictions. In the analysis below, we denote by τ_0 the firm's stochastic liquidation time and consider that investment is at least partially irreversible in that $\varphi < 1$. This partial irreversibility may arise for example from transaction costs, from installation (and des-installation) costs, or from the firm-specific nature of capital.

Management acts in the best interests of shareholders and seeks to maximize shareholder wealth when making policy choices. In the model, management selects not only the firm's investment

policy but also its payout, cash management, financing, and liquidation policies. Notably, we allow management to retain (part of) the firm's earnings inside the firm and denote by C_t the amount of cash that the firm holds at any time t, i.e. its cash buffer. (In the following, we use indifferently the terms cash buffer, cash holdings, and cash inventory.) Cash holdings earn a constant rate of interest $r < \rho$ inside the firm and can be used to fund capital expenditures or to cover operating losses if other sources of funds are costly and/or unavailable. The difference between ρ and r can be interpreted as a carry cost of cash.[3] This cost implies that it is optimal for the firm to start paying dividends when its cash buffer becomes too large.

The firm can increase its cash holdings either by retaining earnings or by raising funds in the capital markets. A key difference between our setup and previous contributions is that we consider that it takes time to secure outside financing and that capital supply is uncertain. In particular, if the firm decides to increase its cash buffer or to finance the capital expenditure by raising outside funds, then it has to search for investors.[4] In the analysis below, we assume that conditional on searching the firm meets outside investors at the jump times of a Poisson process N with arrival rate $(\lambda_t)_{t\geq 0}$. Under these assumptions, the dynamics of the firm's cash reserves are given by

$$dC_t = (rC_{t-} + \mu_0)dt + \sigma dB_t - dD_t + f_{t-}dN_t - 1_{\{t=T\}}K + 1_{\{T\leq t\}}(\mu_1 - \mu_0)dt,$$

where T is the time of investment, f is a nonnegative process that represents the funds raised by the firm upon finding investors, and D is an increasing process with initial value $D_{0-} = 0$ that represents the cumulative dividends paid to shareholders. The firm's cash inventory thus grows with earnings, with outside financing, and with the interest earned on the cash inventory, and decreases with payouts to shareholders and with the cost of investment. As we show below, this specification is flexible enough to accommodate both the case of private firms and the case of large, publicly traded firms for which capital supply may be temporarily limited during a financial crisis or an economic downturn.

[3] The cost of holding cash includes the lower rate of return on these assets because of a liquidity premium and tax disadvantages (Graham (2000) finds that cash retentions are tax-disadvantaged because corporate tax rates generally exceed tax rates on interest income). This cost of carrying cash may also be related to a free cash flow problem within the firm, as in Décamps, Mariotti, Rochet, and Villeneuve (2008) and Bolton, Chen, and Wang (2010).

[4] A growing body of literature argues that assets prices may be more sensitive to supply shocks than standard asset pricing theory would predict. Search theory has played a key role in the formulation of models capturing this idea (see e.g. Duffie, Garleanu, and Perdersen, 2005, Vayanos and Weill, 2008, or Lagos and Rocheteau, 2010). Duffie (2010) provides an early survey of this literature.

As documented by a series of recent empirical studies, capital supply conditions are very important in determining firms' financing decisions, the level of cash holdings, and the level of corporate investment (see the references in Footnote 1 above). These studies also show that firms often face uncertainty regarding their access to capital markets and that this uncertainty has important feedback effects on their policy choices. Our model captures this important feature of capital markets with the stochastic process $(\lambda_t)_{t\geq 0}$, that governs the arrival rate of investors (in general, this arrival rate may depend on both firm characteristics and the supply of funds in capital markets). In the following, we start by analyzing a model in which the arrival rate of investors is constant, given by $\lambda > 0$, so that the probability of finding investors over each time interval $[t, t + dt]$ is λdt and the expected financing lag is $1/\lambda$ (years). Section 5 considers an environment with time-varying capital supply, thereby allowing a study of the effects of capital supply uncertainty for large firms, and shows that our main economic results are unaffected by this extension.

A comparison with some special cases to our setup illustrates how capital supply uncertainty affects firm value and corporate policy choices. When $\lambda = 0$, firms cannot raise funds in the capital markets and have to rely exclusively on internal funds to cover operating losses and to finance capital expenditures. This is the environment considered for example in Radner and Shepp (1996), Décamps and Villeneuve (2007), and Asvanunt, Broadie, and Sundaresan (2007). By contrast, when $\lambda \to \infty$, capital markets are frictionless and firms can instantly raise funds from the financial markets whenever optimal to do so. In that case, the firm has *no need for a cash buffer* and finances both operating losses and capital expenditures by (costlessly) issuing new equity. This is the environment considered for example in Manso (2008), Tserlukevich (2008), Morellec and Schuerhoff (2010), or Carlson, Fisher and Giammarino (2010).

REMARK: While our model could be interpreted as standard investment model with a stochastic investment lag, we show below that the effects of capital supply uncertainty on the firm's optimal policy choices are markedly different from those of investment lags (see e.g. the extension of Dixit, 1989, by Bar-Ilan and Strange, 1996). Most notably, with investment lags and access to outside capital, it is optimal for the firm to distribute all earnings and there is no role for cash holdings.

With capital supply uncertainty, the problem of management is to maximize the present value of future dividends to incumbent shareholders:

$$E_c\left[\int_0^{\tau_0} e^{-\rho t}\left(dD_t - f_{t-}dN_t\right) + e^{-\rho\tau_0}\,\left(\ell_0 + 1_{\{\tau_0>T\}}(\ell_1 - \ell_0)\right)\right],$$

by choosing appropriately the firm's dividend (D), financing (f), and investment (T) policies. The first term in this expression represents the present value of dividend payments to incumbent shareholders until the liquidation time τ_0, net of the claim of new (outside) investors on future firm cash flows. The second term represents the present value of the cash flow to shareholders in liquidation (which depends on whether liquidation occurs before or after investment). Because management optimizes dividend policy and can always decide to pay a liquidating dividend, liquidation will occur when the cash buffer reaches 0. As a result, management only needs to optimize over D and T. In what follows, we denote by $V : [0, \infty) \to [0, \infty)$ the value function of this problem.

3 Value of the firm with no growth option

To facilitate the analysis of the firm's optimization problem, we start by deriving the value of the firm when there is no growth option and the cash flow mean is μ_i, denoted by $V_i(c)$. This function also gives firm value after investment if we set the cash flow mean to μ_1.

When there is no growth option, the firm can follow one of three strategies: (1) pay dividends, (2) retain earnings and search for outside funds, or (3) liquidate. In order to solve the firm's optimization problem, we conjecture (and later verify) that there exists a threshold C_i^* for cash holdings such that the value-maximizing dividend and financing policies can be described as follows:

(a) When $c \leq C_i^*$ the firm should retain earnings, search for outside investors and increase cash holdings to the level C_i^* upon finding investors;

(b) When $c > C_i^*$ the firm should distribute all cash holdings in excess of C_i^*.

We shall now prove this result. Since the firm's initial cash holdings can be above the threshold C_i^*, the value of the firm under the conjectured strategy is given by:

$$V_i(c) = c - C_i^* + V_i(C_i^*), \text{ for } c > C_i^*, \tag{1}$$

implying that it is optimal to distribute all cash holdings above C_i^* with a specially designated dividend or a share repurchase. Below the threshold C_i^*, the optimal policy is to retain earnings and the value of the firm with no growth option satisfies the ordinary differential equation (ODE):

$$\rho V_i(c) = V_i'(c)(rc + \mu_i) + \frac{\sigma^2}{2} V_i''(c) + \lambda \left[V_i(C_i^*) - C_i^* + c - V_i(c) \right]. \tag{2}$$

The left-hand side of equation (2) represents the required rate of return for investing in the firm. The right-hand side is the expected change in firm value in the region where the firm does not pay dividends. The first term captures the effects of cash savings on firm value. The second term captures the effects of cash flow volatility on firm value. The third term reflects the effects of capital supply uncertainty on firm value. This last term is the product of the probability of obtaining outside funds, λ, and the surplus accruing to shareholders when raising the cash buffer from its current level c to its optimal level C_i^*.

The above ODE describes the dynamics of firm value when it is optimal to retain earnings and search for outside funds. When the value of the cash buffer becomes too large, it is optimal to start paying dividends. Similarly, when the cash buffer reaches zero, the firm is liquidated. We thus have the following boundary conditions:

$$V_i(0) = \ell_i, \tag{3}$$

$$\lim_{c \uparrow C_i^*} V_i(c) = V_i(C_i^*), \tag{4}$$

$$\lim_{c \uparrow C_i^*} V_i'(c) = 1, \tag{5}$$

$$\lim_{c \uparrow C_i^*} V_i''(c) = 0. \tag{6}$$

The first boundary condition reflects the fact that the liquidation value of the firm's assets is $\ell_i = \frac{\varphi \mu_i}{\rho}$ and that liquidation occurs when the cash buffer reaches zero. The second condition requires the value function in the retention region to merge with its value at the level of the cash buffer C_i^* where the firm starts paying dividends. The third boundary condition reflects the fact that the firm distributes all cash holdings beyond C_i^* in a minimal way, implying that the marginal value of cash holdings at that point is 1. The last condition is a high-contact condition that allows us to determine the value maximizing payout threshold C_i^*.

To describe the solution to the firm's problem, we need to introduce the following notation. Let

$$F_i(x) = M(-0.5\nu; 0.5; -(rx + \mu_i)^2/(\sigma^2 r)), \tag{7}$$

$$G_i(x) = \frac{rx + \mu_i}{\sigma\sqrt{r}}\ M(-0.5(\nu - 1); 1.5; -(rx + \mu_i)^2/(\sigma^2 r)), \tag{8}$$

where $\nu = (\rho + \lambda)/r$ and M is the confluent hypergeometric function (see Abramowitz and Stegun, 1970, Chapter 15). Solving the firm's problem yields the following result:

Proposition 1 *There exists a unique level for the cash buffer C_i^* that maximizes the value V_i of a firm with no growth option, for $i = 0, 1$. This optimal cash buffer is the unique solution to*

$$\alpha_i(C_i^*)F_i(0) - \beta_i(C_i^*)G_i(0) + \frac{\lambda}{\rho+\lambda}\left(\frac{(r-\rho)C_i^* + \mu_i}{\rho} + \frac{\mu_i}{\rho+\lambda-r}\right) = \ell_i$$

where the functions $\alpha_i(c)$ and $\beta_i(c)$ are defined by

$$\alpha_i(c) = \frac{-G_i''(c)(\rho-r)}{2\sigma^{-3}\sqrt{r}(\rho+\lambda-r)(\rho+\lambda)e^{-(\sigma^2 r)^{-1}(rC+\mu_i)^2}}, \tag{9}$$

$$\beta_i(c) = \frac{-F_i''(c)(\rho-r)}{2\sigma^{-3}\sqrt{r}(\rho+\lambda-r)(\rho+\lambda)e^{-(\sigma^2 r)^{-1}(rC+\mu_i)^2}}. \tag{10}$$

For any level of the cash buffer $c < C_i^*$, the value of a firm with no growth option is

$$V_i(c) = \alpha_i(C_i^*)F_i(c) - \beta_i(C_i^*)G_i(c) + \frac{\lambda}{\rho+\lambda}\left(V_i(C_i^*) + c - C_i^* + \frac{\mu_i + rc}{\rho+\lambda-r}\right), \tag{11}$$

where firm value at the optimal cash buffer satisfies

$$V_i(C_i^*) = \frac{rC_i^* + \mu_i}{\rho}.$$

The expression for the value of the firm in Proposition 1 can be interpreted as follows. The first two terms of equation (11) represent the present value of the cash flows accruing to shareholders when cash holdings reach 0, at which point it is optimal to liquidate the firm's assets, or when they reach C_i^*, at which point it is optimal to start paying dividends. The last term on the right hand side of equation (11) reflects the effects on firm value of the change in the cash buffer due to the arrival of outside investors. In particular, we have

$$E_c\left[e^{-\rho\theta}\left(V_i(C_i^*) - C_i^* + C_\theta\right)\right] = \frac{\lambda}{\rho+\lambda}\left(V_i(C_i^*) + c - C_i^* + \frac{\mu_i + rc}{\rho+\lambda-r}\right)$$

where θ is the (random) time at which the firm raises capital from outside investors and increases its cash holdings from their current level c to the optimal level C_i^*.

Proposition 1 shows that, in line with DeAngelo, DeAngelo, and Stulz (2006), the firm starts paying dividends when retained earnings reach a performance threshold C_i^*. In order to better understand the strategy of the firm, Figure 1 plots C_i^* as a function of the arrival rate of investors λ, the reinvestment rate r, the recovery rate on assets φ, and cash flow volatility σ. The base parametrization in this figure is $\rho = .06$, $r = .05$, $\lambda = 4$, $\sigma = .1$, $\mu = .1$, and $\varphi = .75$, implying a haircut of 25% of asset value in liquidation, an expected financing lag of $1/\lambda = 3$ months, and a cost of holding cash of 1% per year.

Consistent with economic intuition, the figure shows that the optimal level of cash holdings decreases with the arrival rate of investors (see Lemma 17 in the Appendix). That is, as λ increases, the likelihood of finding outside investors to cover operating losses increases and the need to hoard cash within the firm decreases. The figure also demonstrates that even for large values of the arrival rate of investors, the firm still optimally carries a significant cash buffer. These results are consistent with the evidence in Opler, Pinkowitz, Stulz and Williamson (1999) and Bates, Kahle, and Stulz (2009), who find that firms hold more cash when their access to external capital markets is more limited.

Insert Figure 1 Here

Another property of the firm's optimal policy illustrated by the figure is that cash holdings should increase with cash flow volatility. This result is consistent with the evidence in Harford (1999) and Bates, Kahle, and Stulz (2009). It also suggests that the increase in cash holdings over the 1980-2005 period can be explained by the increase in idiosyncratic volatility reported by Irvine and Pontiff (2009). As expected, the figure also shows that cash reserves increase when the opportunity cost of holding cash decreases (i.e. when r increases).

Finally, as illustrated by the figure, another key determinant of the optimal cash buffer is the liquidation value of assets (or the degree of irreversibility of investment). In particular, the optimal level of cash holdings is monotonically decreasing in φ and converges to zero as φ tends to one (see the Appendix). One direct testable prediction of the model is that firms with more tangible assets should have lower cash holdings. This prediction of the model is novel, and provides grounds for further empirical work on the determinants of cash holdings.

4 Firm value with a growth option

4.1 Firm value and optimal cash holdings

We now turn to the analysis of firm value when management has the option to increase earnings by paying a lump sum cost K. The growth option changes the firm's policy choices and firm value only if the project has positive net present value (NPV). The following proposition provides a necessary and sufficient condition for this to be the case.

Proposition 2 *The option to invest has positive NPV if and only if the cost of investment K is below K^* defined by:*

$$\frac{\mu_1 - \mu_0}{\rho} = K^* + \left(1 - \frac{r}{\rho}\right)(C_1^* - C_0^*),$$

where C_i^* is the value-maximizing cash inventory for a firm with assets in place that deliver a cash flow with mean μ_i and with no growth option.

The intuition for this result is straightforward. The left hand side of equation (2) represents the expected present value of the increase in firm cash flows following the exercise of the growth option. The right hand side represents the total cost of investment, which incorporates the direct cost of investment and the change in the cost of carrying the cash balance. In the following, we consider that the cost of investment is below K^* defined by equation (2) so that the value of the firm's growth option is positive.

To solve the firm's optimization problem in the presence of the growth option, consider the following two alternative strategies.

(W) The firm finances investment exclusively with external funds and retains earnings until the cash buffer reaches C_W^*, at which point it starts paying dividends.

(U) The firm finances investment with external or internal funds, retains earnings, and invests in the growth option when the cash buffer reaches a level an optimally determined level C_U^* or upon finding investors.

Let $W(c)$ denote the value of the firm under the first strategy and $U(c)$ denote the value of the firm under the second strategy. Using standard arguments, it is immediate to show that in the continuation region where it is optimal to retain earnings $W(c)$ and $U(c)$ satisfy the following ODE:

$$\rho J(c) = J'(c)(rc + \mu_0) + \frac{\sigma^2}{2} J''(c) + \lambda \left(V_1(C_1^*) - C_1^* - K + c - J(c)\right) \tag{12}$$

for $J = W,\ U$. The left hand side of this equation is again the return required by investors for investing in the firm. The right hand side is the expected change in firm value due to the effects of a change in cash savings (first term), cash flow volatility (second term), and the arrival of outside investors (third term). Since the firm invests in the project and readjusts its cash buffer C_1^* upon

finding investors, the change in the value of incumbent shareholders due to the arrival of outside investors is given by $V_1(C_1^*) - C_1^* - K + c - J(c)$.

These ODEs are solved subject to the following boundary conditions:

$$W(0) = U(0) = \ell_0, \tag{13}$$

$$W(c) = W(C_W^*) + c - C_W^*, \text{ for } c \geq C_W^*, \tag{14}$$

$$\lim_{c \uparrow C_W^*} W'(c) = 1, \tag{15}$$

$$\lim_{c \uparrow C_W^*} W''(c) = 0, \tag{16}$$

$$U(c) = V_1(c - K), \text{ for } c \geq C_U^*, \tag{17}$$

$$U'(C_U^*) = V_1'(C_U^* - K). \tag{18}$$

Condition (13) requires firm value to be equal to ℓ_0 in the absence of cash reserves since the firm liquidates at $c = 0$. Condition (14) reflects the fact that it is optimal to make a payment $c - C_W^*$ to shareholders whenever cash holdings are above C_W^*. Condition (15) follows from condition (14) and reflects the fact that the firm distributes all cash holdings beyond C_W^* in a minimal way, implying that the marginal value of cash holdings at that point is 1. Condition (16) is a high-contact condition that allows us to determine the value maximizing payout threshold C_W^*. Condition (17) reflects the fact that it is optimal to invest with internal funds whenever the cash buffer exceeds C_U^*. Finally, condition (18) is a smooth-pasting (optimality) condition that allows us to determine the value-maximizing exercise trigger C_U^*.

Solving for $U(c)$ and $W(c)$ yields the following proposition.

Proposition 3 *Assume that $K \leq K^*$ so that the growth option has positive NPV. Then the firm values associated with the strategies (W) and (U) are respectively given by*

$$W(c) = \begin{cases} \alpha_0(C_W^*)F_0(c) - \beta_0(C_W^*)G_0(c) + \Phi(c), & c \leq C_W^*, \\ c - C_W^* + W(C_W^*), & \textit{otherwise}, \end{cases} \tag{19}$$

and

$$U(c) = \begin{cases} \xi_G(C_U^*)F_0(c) - \xi_F(C_U^*)G_0(c) + \Phi(c), & c \leq C_U^*, \\ V_1(c - K), & \textit{otherwise}, \end{cases} \tag{20}$$

where

$$\Phi(c) = \frac{\lambda(V_1(C_1^*) - (C_1^* + K))}{\rho + \lambda} + \frac{\lambda(\mu_0 + (\rho + \lambda)c)}{(\rho + \lambda)(\rho + \lambda - r)}, \tag{21}$$

the constants C_W^* *and* C_U^* *are the unique solutions to*

$$\alpha_0(C_W^*)F_0(0) - \beta_0(C_W^*)G_0(0) = \xi_G(C_U^*)F_0(0) - \xi_F(C_U^*)G_0(0) = \ell_0 - \Phi(0), \tag{22}$$

and the functions ξ_F *and* ξ_G *are defined in the Appendix.*

Having characterized firm value for the strategies (U) and (W), we now return to management's optimization problem. Intuitively, we expect the firm to follow strategy (U) when the cost of investment is low since in that case the cost of building up the cash buffer to the level required for investment is low, independently of the current level of cash holdings. By contrast, we expect the firm to adapt its strategy to the level of its cash holdings when the cost of investment is high. In particular, the firm should follow strategy (W) when its cash holdings are below a certain threshold C_L^*, since in that case it would be too costly to build up the cash buffer to invest with internal funds. Above the threshold C_L^*, the firm should retain earnings and invest either when its cash buffer reaches $C_H^* \geq C_U^*$ or when outside financing arrives.

Accordingly, we have that for high investment costs (i.e., for $K > K^{**}$ defined in Theorem 4 below), firm value is given by $V(c) = W(c)$ for $c \leq C_L^*$, by $V(c) = V_1(c - K)$ for $c \geq C_H^*$, and satisfies the ODE:

$$\rho V(c) = V'(c)(rc + \mu_0) + \frac{\sigma^2}{2}V''(c) + \lambda\left(V_1(C_1^*) - C_1^* - K + c - V(c)\right), \tag{23}$$

between the thresholds C_L^* and C_H^* . This ODE is solved subject to the boundary conditions

$$V(C_L^*) = W(C_L^*), \tag{24}$$

$$V(C_H^*) = V_1(C_H^* - K), \tag{25}$$

$$V'(C_L^*) = W'(C_L^*), \tag{26}$$

$$V'(C_H^*) = V_1'(C_H^* - K). \tag{27}$$

Condition (24) requires firm value to coincide with W at the point C_L^* where the firm switches to strategy (W). Condition (25) requires firm value to be equal to the payoff from investment when investing with internal funds at the point $C_H^* \geq C_U^*$. Finally, conditions (26) and (27) are smooth pasting conditions that allow us to determine the optimal switching points C_L^* and C_H^*.

Solving management's optimization problem yields our main result.

Theorem 4 *There exist two thresholds K^{**} and K^* for the cost of investment, with $0 < K^{**} < K^*$, such that*

(a) *If $K \geq K^*$, the growth option is worthless and optimal policy choices are as in Proposition 1.*

(b) *If $K < K^{**}$, firm value is given by $V(c) = U(c)$ and the optimal policy is to retain earnings and to invest in the growth option when the cash buffer reaches $C_U^* \leq C_1^* + K$ or when outside financing arrives.*

(c) *If $K^{**} < K < K^*$, firm value is given by*

$$V(c) = \begin{cases} W(c), & c \leq C_L^*, \\ S(c), & C_L^* \leq c \leq C_H^*, \\ V_1(c-K), & \text{otherwise} \end{cases}$$

where the function $S(c)$ is defined by

$$S(c) = \xi_G(C_H^*)F_0(c) - \xi_F(C_H^*)G_0(c) + \Phi(c) \tag{28}$$

and the constants $C_L^* \geq C_W^*$ and $C_H^* \in [C_U^*, C_1^* + K]$ are the unique solutions to (24) and (25). When $c \leq C_W^*$, the optimal policy is to invest exclusively with outside funds and to retain earnings until the cash buffer reaches C_W^*. When $c \in [C_W^*, C_L^*]$, the optimal policy is to make a lump sum payment $c - C_W^*$ and then to follow the optimal policy for $c \leq C_W^*$. When $c > C_L^*$, the optimal policy is to build up the cash buffer and exercise the option either when the cash buffer reaches C_H^* or when outside financing arrives.

(d) *If $K = K^{**}$ the firm is indifferent between strategies (U) and (W).*

Theorem 4 provides a complete characterization of the firm's optimal policy choices and of firm value under these policies. Several important results follow from this theorem. First, the theorem shows that when $K < K^{**}$, the firm may use its cash buffer both to cover operating losses and to finance investment. Such situations arise when the cost of hoarding cash inside the firm is not too high or when the NPV of the project is large. We show below that, while cash holdings serve in principle two purposes in this case, the probability of financing investment with cash holdings is low, implying that cash holdings essentially represent a risk management tool aimed at insuring the firm against potential losses.

Second, Theorem 4 shows that when $K^{**} \leq K < K^*$, the optimal strategy for the firm depends on the current level of cash reserves. In particular, when cash reserves are below C_W^*, the optimal policy is to to finance the capital expenditure with external funds and to use cash reserves to cover operating losses. When cash reserves are between C_W^* and C_L^*, the firm's dividend policy will consist in paying both regular dividends and specially designated dividends (or making share repurchases). When cash holdings are above C_L^*, the firm can finance the capital expenditure using internal or external funds and the optimal policy is to retain earnings until the firm invests in the project. Interestingly, the change in financing technology occurring at C_L^* makes the value-function locally convex. This implies that our financially constrained firm may want to engage in risk increasing strategies at this point, in contrast to the results in Froot, Scharfstein, and Stein (1993).

When financing the growth option with external funds, the value-maximizing policy for the firm is to raise enough funds to finance both the capital expenditure and the potential gap between current cash holdings and the optimal level after investment C_1^*. That is, firms always increase their cash buffer when raising funds from outside investors. This prediction of the model is consistent with the evidence in Kim and Weisbach (2008) and McLean (2010), who find that firms' decisions to issue equity are essentially driven by their desire to build up cash reserves. By contrast, when financing the capital expenditure internally, the optimal policy is to invest at a level of cash holdings below $C_1^* + K$, implying that cash holdings are below their optimal level after investment. This effect is due to the positive discounting and to the fact that hoarding cash inside the firm is costly.

4.2 Model implications

4.2.1 Cash holdings before investment

When the firm has a growth option, cash holdings serve two purposes. First, they can be used to cover unexpected operating losses. Second, they can be used to finance investment. In order to analyze the effects of this second motive on the value-maximizing level of the cash buffer, consider an economic environment in which $K < K^{**}$ so that $V(c) = U(c)$. In such environments, the firm's cash holdings are in $(0, C_U^*)$ before investment and the difference between C_U^* and C_0^* represents the change in the optimal cash buffer due to the growth option.

Figure 2, Panel A, plots the optimal cash buffer C_U^* as a function of the arrival rate of investors λ, the recovery rate on assets φ, cash flow volatility σ, and the growth potential of the firm $\mu_1 - \mu_0$.

Figure 2, Panel B, plots the change in the optimal cash buffer $C_U^* - C_0^*$ due to the growth option as a function of these same parameters. In this figure, we use the same parameter values as in Figure 1 and set $K = 0.2$ and $\mu_1 = 0.125$ (so that operating cash flows increase by 25% upon investment).

Insert Figure 2 Here

Several results follow from Figure 2. First the figure shows that C_U^* exhibits the same sensitivity to the input parameters of the model as C_0^*. That is, optimal cash holdings increase with cash flow volatility σ and decrease with the arrival rate of investors λ and with the recovery rate of assets in liquidation φ. (Note that C_U^* converges to K as φ tends to 1 since the firm wants to be able to finance the capital expenditure with internal funds.)

Second, the figure reveals that most often the optimal level of cash holdings before investment exceeds that after investment, in that $C_U^* > C_0^*$. This is due to the fact that when the cash buffer reaches C_U^*, the firm needs to finance the capital expenditure out of internal funds, thereby increasing the probability of inefficient liquidation after investment. Interestingly, the figure shows that C_U^* can be lower than C_0^* when there is a large benefit to investment (i.e. when $\mu_1 - \mu_0$ is large) or when the risk of not investing is important (i.e. when σ is high). That is, the optimal level of cash holdings can be lower when the firm has two motives for holding cash. This effect is mitigated by a stronger supply of capital since the risk of being unable to cover operating losses or to finance investment decreases with λ.

4.2.2 Financing investment

An important question is whether capital supply uncertainty actually affects growth and the source of financing used by firms when investing. To answer this question, consider again an economic environment in which $K < K^{**}$ so that $V(c) = U(c)$. In such economic environments, the firm invests with either internal funds or with outside capital. The probability that the firm invests using internal funds is (for $c \leq C_U^*$):

$$P_I(c) = P_c[\tau_U \leq \theta \wedge \tau_0] = E_c\left[\int_0^\infty \lambda e^{-\lambda t} 1_{\{\tau_U \leq t \wedge \tau_0\}} dt\right] = E_c\left[1_{\{\tau_U \leq \tau_0\}} e^{-\lambda \tau_U}\right]$$

where τ_U is the first time that the cash reserve process

$$C_t = e^{rt} C_0 + \frac{\mu_0}{r}\left(e^{rt} - 1\right) + \int_0^t e^{r(t-s)} \sigma dB_s$$

is equal to $C_U^* \geq 0$, and θ is the first time at which external financing becomes available. Similarly, the probability that the firm invests using external funds is

$$P_E(c) = P_c[\theta \leq \tau_U \wedge \tau_0] = 1 - P_c[\theta > \tau_U \wedge \tau_0] = 1 - E_c\left[e^{-\lambda(\tau_U \wedge \tau_0)}\right].$$

Using standard arguments together with our previous results and notation, it is immediate to establish the following result:

Proposition 5 *The probabilities that the firm invests with internal or external funds are respectively given by:*

$$P_I(c) = \alpha_I F(c) + \beta_I G(c),$$
$$P_E(c) = 1 - \alpha_E F(c) - \beta_E G(c),$$

where the functions F, G are defined by

$$F(c) = F_0(c)|_{\rho=0},$$
$$G(c) = G_0(c)|_{\rho=0},$$

with F_0 and G_0 defined by equations (7) and (8) and where the constants α_E, β_E, α_I, β_I solve the following system of equations

$$\alpha_I F(0) + \beta_I G(0) = 0,$$
$$\alpha_E F(0) + \beta_E G(0) = 1,$$
$$\alpha_j F(C_U^*) + \beta_j G(C_U^*) = 1, \text{ for } j = I, E$$

Using Proposition 5 we can examine the financing strategy of firms when investing in the growth option. In particular, Figure 3 plots the probability of investment using internal funds (dashed line) and the probability of investment using external funds (solid line) as functions of the arrival rate of investors λ, the recovery rate on assets φ, cash flow volatility σ, and the growth potential of the firm $\mu_1 - \mu_0$. In this figure, we use the same parameter values as in Figure 1 and set the additional parameters as follows: $C_0 = K = 0.2$ and $\mu_1 = 0.125$.

Insert Figure 3 Here

Consistent with economic intuition, the figure shows that as the arrival rate of investors increases, the probability of financing the capital expenditure with external funds increases. The

figure also shows that the arrival rate of investors has a very important effect on these probabilities. For example, when $\lambda = 4$ (implying an expected financing lag of 3 months), the probability of investment with internal funds is 12%. When $\lambda = 12$ (implying an expected financing lag of 1 month), the probability of investment with internal funds is 3%. This result suggests that in most economic environments, cash holdings will be used mostly to cover operating losses, consistent with the evidence in the large sample studies by Opler, Pinkowitz, Stulz and Williamson (1999) and Bates, Kahle, and Stulz (2009) and in the survey of Lins, Servaes, and Tufano (2010).

Another interesting property of the model illustrated by Figure 3 is that the probability of investment with internal funds increases with the liquidation value of assets. This is a direct consequence of the fact that the investment threshold C_U^* decreases with φ. Also, as volatility increases, the firm wants to hold more cash to prevent liquidation and the probability of liquidation increases. These two effects imply that the probability of investment with internal funds decreases with σ. By contrast, the probability of investment with external funds first increases (the first effect dominates) and then decreases (the second effect dominates) with σ.

To investigate further the effects of capital supply uncertainty on investment and liquidation probabilities, Figure 4 plots the probability of investing with internal funds, the probability of investing with external funds, and the probability of liquidation over a one-year (dashed line) and over a three-year (solid line) horizon as functions of the arrival rate of investors λ. In this figure, we use the same input parameter values as in Figure 3.

Insert Figure 4 Here

Several interesting properties of the model are illustrated by Figure 4. First, the probability of investment with external funds monotonically increases with the arrival rate of investors while the probability of investment with internal funds and the probability of liquidation monotonically decrease with this rate. Indeed, an increase in λ results in a decrease in the investment threshold with internal funds and in an increase of the matching rate between the firm and investors. Second, and related to the above, the overall probability of investment decreases as λ decreases, implying that a negative shock to the supply of capital may hamper investment even if firms have enough financial slack to fund all profitable investment opportunities internally.

5 Extensions

In this section, we consider two important extensions of our model. First, we introduce issuance costs when the firm raises outside funds. Second, we introduce time-varying capital supply by allowing the arrival rate of investors to change stochastically through time. For simplicity, we consider that the firm only has assets in place and denote the growth rate of cash flows by μ.

5.1 Issuance costs

There is considerable evidence that firms have to pay significant costs when issuing securities and that these costs exhibit economies of scale (see Smith (1977) for an early survey and Altinkilic and Hansen (2000) or Kim, Palia, and Saunders (2008) for recent evidence). To capture this important feature of capital markets, we consider that when raising outside funds, the firm has to pay a proportional cost ε and a fixed cost κ. This implies that if the firm raises an amount $A(1+\varepsilon)+\kappa$ from the financial markets, it gets A. Let $V(c) = V(c,\kappa,\varepsilon)$ be the corresponding value function. For a given level of cash holdings c, the firm will find it optimal to raise outside funds if

$$\max_{f\geq 0}\left[V(c+f)-(1+\varepsilon)f\right] - \kappa > V(c)\,.$$

Consider first the effects of proportional costs on the decision to raise outside funds. If firm value V_i is a concave function of cash holdings (which we establish below), then there exists a threshold $\overline{C} < C^*$ such that $V'(\overline{C}) = 1+\varepsilon$. Outside funds will then never be raised when $c > \overline{C}$ since in that case the marginal cost of outside funds is larger than their marginal benefit. The firm will therefore only raise funds when its cash holdings are below $\overline{C}$ to bring its cash buffer to $\overline{C}$, where the marginal cost and benefits of outside funds are equalized.

Consider next the effects of fixed costs. Clearly, the firm will raise outside funds only if it is profitable to do so, that is if the current cash buffer is such that

$$V(\overline{C}) - (1+\varepsilon)\,\overline{C} > V(c) - (1+\varepsilon)\,c + \kappa\,.$$

Since $V'(c) \geq 1+\varepsilon$ for $c \leq \overline{C}$, we get that $V(c) - (1+\varepsilon)\,c$ is monotone increasing for $c \leq \overline{C}$. Therefore, there exists a threshold for cash holdings $\underline{C} < \overline{C}$ satisfying

$$V(\overline{C}) - (1+\varepsilon)\,\overline{C} = V(\underline{C}) - (1+\varepsilon)\,\underline{C} + \kappa\,,$$

such that outside funds are raised if and only if $c \leq \underline{C}$.

We are now in a position to determine the value of the firm in the presence of issuance costs. Specifically, using the same steps as above, it is immediate to show that, in the region where it is optimal to retain earnings, firm value V satisfies the ODE:

$$\rho V(c) = V'(c)(rc+\mu) + \frac{\sigma^2}{2}V''(c) + 1_{\{c\leq \underline{C}\}}\,\lambda\left[V(\underline{C}) - (1+\varepsilon)\,\underline{C} + (1+\varepsilon)\,c - V(c)\right]. \tag{29}$$

In particular, in the region where outside funds are never raised, we have that firm value satisfies

$$\rho V(c) = V'(c)(rc+\mu) + \frac{\sigma^2}{2}V''(c)\,. \tag{30}$$

Denote by $V_{\lambda=0}(c)$ the value function when outside financing is not available (defined in equation (11) with $\lambda = 0$) and by $C^*_{\lambda=0}$ the corresponding optimal level of cash holdings. We can find upper bounds for the proportional and fixed issuance costs by examining whether it is optimal for the firm to raise outside funds when the cash balance reaches zero, i.e. when the benefits of outside funds are the largest. Assuming that there are no fixed costs, it will be optimal for the firm to raise outside funds if proportional costs ε are below $\overline{\varepsilon}$ defined by

$$\overline{\varepsilon} \;=\; V'_{\lambda=0}(0) \;-\; 1\,. \tag{31}$$

By concavity, if proportional costs satisfy $\varepsilon < \overline{\varepsilon}$, then there exists a level of cash holdings $\overline{C}_{\lambda=0}$ such that $V'_{\lambda=0}(\overline{C}_{\lambda=0}) \;=\; 1+\varepsilon$ and the maximum level of fixed costs that makes it optimal for the firm to raise outside funds, denoted by $\overline{\kappa}(\varepsilon)$, is then defined by

$$\overline{\kappa}(\varepsilon) \;=\; V_{\lambda=0}(\overline{C}_{\lambda=0}) - (1+\varepsilon)\overline{C}_{\lambda=0} \;-\; \ell\,. \tag{32}$$

This leads to the following result.

Proposition 6 *It is optimal for the firm to raise funds from outside investors if and only if the proportional and fixed issuance costs satisfy $\varepsilon < \overline{\varepsilon}$ and $\kappa < \overline{\kappa}(\varepsilon)$, where $\overline{\varepsilon}$ and $\overline{\kappa}(\varepsilon)$ are defined in equations (31) and (32). In this case, there exist three positive thresholds $\underline{C} < \overline{C} < C^*$ for the firm's cash holdings such that the optimal policy consists in distributing dividends whenever the cash buffer reaches C^*, and to raises outside funds to reach the level $\overline{C}$ only when $c < \underline{C}$.*

In the following, we consider that the proportional and fixed issuance costs satisfy $\varepsilon < \overline{\varepsilon}$ and $\kappa < \overline{\kappa}(\varepsilon)$, so that it may be optimal for the firm to raise funds from outside investors. The value of the firm is then the unique solution to equation (29) satisfying the boundary conditions $V(0) = \ell$, $V'_i(C^*) \;=\; 1$, and $V''(C^*) \;=\; 0$, which describe the value function at the liquidation and dividend payment triggers. Solving management's optimization problem yields the following:

Proposition 7 *There exists a unique level for the cash buffer C^* that maximizes firm value when it is costly to raise outside funds. For any level of the cash buffer $c < C^*$, firm value satisfies*

$$V(c) \;=\; \begin{cases} V(c;C^*)\;, & c \le \underline{C} \\ V_{\lambda=0}(c;C^*)\;, & c > \underline{C}\,. \end{cases}$$

where

$$V(c;C^*) \;= a(C^*)\,F(c) + b(C^*)\,G(c) + \frac{\lambda}{\rho+\lambda}\left[V_{\lambda=0}(\underline{C};C^*) + (1+\varepsilon)\,(c-C^*) + \frac{\mu+rc}{\rho+\lambda-r}\right] \quad (33)$$

and $V_{\lambda=0}(c,C^*)$ is defined in equation (11) with $\lambda = 0$. In these equations, the optimal cash buffer is the unique solution to

$$\frac{\varphi\mu}{\rho} \;=\; a(C^*)\,F(0) \;+\; b(C^*)\,G(0) \;+\; \frac{(\rho+\lambda-r)\lambda(V_{\lambda=0}(\underline{C};C^*) - (1+\varepsilon)\underline{C}) + \lambda\mu}{(\rho+\lambda-r)(\rho+\lambda)}\,,$$

where the parameters $a(C^*)$ and $b(C^*)$ are determined using the value-matching and smooth-pasting conditions $V(\underline{C};C^*) = V_{\lambda=0}(\underline{C};C^*)$ and $V'(\underline{C};C^*) = V'_{\lambda=0}(\underline{C};C^*)$, the functions $F(c)$ and $G(c)$ are defined by equations (7) and (8) with $\mu_i = \mu$, and the thresholds $\underline{C} < \overline{C} < C^*$ solve

$$V'_{\lambda=0}(\overline{C};C^*) \;=\; 1+\varepsilon\;, \quad (34)$$

$$V_{\lambda=0}(\overline{C};C^*) \;-\; (1+\varepsilon)\,\overline{C} \;=\; V(\underline{C};C^*) \;-\; (1+\varepsilon)\,\underline{C} \;+\; \kappa\,. \quad (35)$$

The expression for the value of the firm reported in Proposition 7 is similar to that in Proposition 1 and admits a similar interpretation. In particular, the first two terms of equation (33) represent the present value of the cash flows accruing to shareholders when cash holdings reach 0, at which point it is optimal to liquidate the firm, or when they reach C^*, at which point it is optimal to start paying dividends. The last term on the right hand side of equation (33) reflects the effects on firm value of the change in the cash buffer from $c < \underline{C}$ to $\overline{C} < C^*$ due to the arrival of outside investors. Finally, equations (34) and (35) show that the marginal benefit of cash holdings is equal to the marginal cost of raising funds at the threshold $\overline{C}$ and that the increase in firm value due to a change in cash holdings is equal to the total cost of increasing cash holdings at the threshold $\underline{C}$.

One interesting feature of our model is that the firm will find it optimal to raise outside funds even when cash holdings are positive. This is in contrast with the result in Décamps, Mariotti, Rochet, and Villeneuve (2008) or Bolton, Chen, and Wang (2010), in which the firm either never issues securities (when issuance costs are too large) or only raises outside funds when cash holdings

reach zero. In addition, their model predicts that firms are never liquidated in the later case. By contrast our model predicts that firms will raise cash even if they have a positive cash balance and that some firms will be liquidated even if issuance costs are low.

To better understand the strategy for the firm in the presence of issuance costs, Figure 5 plots the optimal level of cash holdings C^* (black line), the refinancing trigger $\underline{C}$ (red line), and the firm's cash holdings after refinancing $\overline{C}$ (blue line) as functions of the proportional cost of issuance ε, the fixed cost of issuance κ, the arrival rate of investors λ, and cash flow volatility σ. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\sigma = .1$, $\mu = .1$, $\varphi = .75$, $\varepsilon = 0.05$, and $\kappa = 0.025$.

Insert Figure 5 Here

Consistent with economic intuition, the figure shows that the optimal level of cash holdings C^* increases with both the proportional cost of issuance ε and the fixed cost of issuance κ. That is, as outside funds become more expensive, it becomes relatively cheaper for the firm to hoard cash making it optimal to delay dividend distributions. The quantitative effect is however limited. As in the model without issuance costs, cash holdings increase with cash flow volatility and decrease with the arrival rate of investors. The figure also shows that proportional issuance costs have a dramatic effect on the firm's cash holdings after refinancing $\overline{C}$ since the cost of outside funds is equalized with the benefits of cash holdings at that point. Overall, the main conclusions from the model seem resilient to the specific parametric assumptions.

5.2 Time-varying capital supply

So far, we have ignored the possibility that the arrival rate of investors, and hence that corporate policy choices, could change over time. The global financial crisis of 2007-2008 has shown however that the supply of external finance for non-financial firms could be subject to significant shocks. This section extends our basic framework to consider shocks to the supply of capital, thereby allowing a study of the effects of capital supply uncertainty for large, publicly traded firms for which access to outside funds may be limited only during a financial crisis or an economic downturn.

To do so, we consider an environment in which the arrival rate of investors $(\lambda_t)_{t\geq 0}$ can take two values: λ_L and λ_H with $\lambda_H > \lambda_L \geq 0$. In addition, we assume that λ_t is observable and that the transition between the two states follows a Poisson law. Let $\pi_{HL} > 0$ denote the rate of leaving

state H and T_H denote the time to leave state H. Within our model, the exponential law holds:

$$\Pr(T_H > t) = e^{-\pi_{HL} t},$$

and there is a probability $\pi_{HL} dt$ that the arrival rate $(\lambda_t)_{t \geq 0}$ changes from λ_H to λ_L during an infinitesimal time interval dt. In this specification, the capital supply regime H corresponds to economic conditions in which capital is readily available while the capital supply regime L corresponds to economic conditions in which capital is scarce.

As in the model with constant arrival rate of investors, the objective of management is to determine the cash holdings and dividend policies that maximize shareholder wealth. One essential difference with the previous formulation however is that because the arrival rate of investors depends on the current regime (high or low supply of capital), so do the value-maximizing trigger for dividend payments and firm value. In other words, when capital supply can shift between two levels there exist two dividend distribution thresholds C_H^* and C_L^* and two value functions V_H and V_L.

Using standard methods (see e.g. Guo, Miao, and Morellec (2005)), it is possible to show that the value function pair (V_L, V_H) satisfies the following set of ODEs:

$$\begin{aligned}
\rho V_L(c) &= V_L'(c)(rc + \mu) + \frac{\sigma^2}{2} V_L''(c) + \lambda_L \left[V_L(C_L^*) - C_L^* + c - V_L(c) \right] + \pi_{LH} \left[V_H - V_L \right] \\
\rho V_H(c) &= V_H'(c)(rc + \mu) + \frac{\sigma^2}{2} V_H''(c) + \lambda_H \left[V_H(C_H^*) - C_H^* + c - V_H(c) \right] + \pi_{HL} \left[V_L - V_H \right]
\end{aligned}$$

These equations are similar to the one obtained in the constant capital supply case. However, they contain an additional term ($\pi_{LH} \left[V_H - V_L \right]$ or $\pi_{HL} \left[V_L - V_H \right]$) that corresponds to the probability weighted change in firm value due to a change in the supply of capital. As before, these equations are solved subject to boundary conditions at the dividend thresholds C_H^* and C_L^* and at zero.

Appendix E provides a full characterization of the solution to management's optimization problem. Using this solution, we can then examine the effects of time-varying capital supply on the firm's policy choices. To this end, Figure 6 plots the optimal levels of cash holdings C_H^* (solid line) and C_L^* (dashed line) as functions of the recovery rate φ, the arrival rate of investors in the low capital supply regime λ_L, cash flow volatility σ and the interest rate r. The base parametrization is $\rho = .06$, $r = .05$, $\lambda_L = 4$, $\lambda_H = 24$, $\pi_{LH} = 1.154$, $\pi_{HL} = 0.223$, $\sigma = .1$, $\mu = .1$, and $\varphi = .75$. The rates of leaving states H and L are calibrated to match the average durations of NBER recessions.

The values of the arrival rates of investors imply that, conditional on staying in a given regime, the expected financing lag is 3 months in regime L and 15 days in regime H.

Insert Figure 6 Here

Consistent with economic intuition, the figure shows that the optimal cash holdings policy takes the form of a trigger policy and that there exists one optimal cash buffer per level of capital supply. Since the two regimes (high and low capital supply) are related to one another through the πs, so do the optimal cash buffers. Specifically, a lower persistence of regime i (i.e. a higher π_{ij}) reduces the opportunity cost of paying dividends in regime i, and hence narrows the gap between the optimal cash buffers in the two capital supply regimes. This effect is illustrated in the figure by the two dotted lines in each panel, where we have set the πs to zero (so that the firm faces only one level of capital supply forever). Finally the figure reveals that even when the capital supply is strong (i.e. when $\lambda_t = \lambda_H$), the firm carries a significant cash buffer as a hedge against a possible change in the supply of capital.

6 Conclusion

Following Modigliani and Miller (1958), extant theoretical research in corporate finance generally assumes that capital markets are frictionless so that corporate behavior and capital availability depend solely on firm characteristics. This demand-driven approach has recently been called into question by a large number of empirical studies. These studies document the central role of supply conditions in capital markets in explaining corporate policy choices and highlight the need for an improved understanding of the precise role of supply.

This paper takes a first step in constructing a dynamic model of corporate investment, payout, cash management, and financing decisions with capital supply effects by considering a setup in which firms face uncertainty regarding their ability to raise funds in the capital markets. The model provides an explicit characterization of corporate policy choices for a firm acting in the best interests of incumbent shareholders and shows that capital market frictions have first-order effects on corporate behavior. In particular, the model shows that

1. Cash holdings should increase with cash flow volatility and decrease with the firm's access to outside capital, in line with Opler, Pinkowitz, Stulz and Williamson (1999), Harford (1999), and Bates, Kahle, and Stulz (2009).

2. Negative shocks to the supply of capital should hamper investment even if firms have enough slack to finance investment internally, consistent with the evidence in the large sample studies by Kashyap, Stein and Wilcox (1993), Gan (2007), Becker (2007), or Lemmon and Roberts (2007), and with the survey of Campello, Graham, and Harvey (2010).

3. Firms should only start paying dividends when retained earnings reach a performance threshold, in line with DeAngelo, DeAngelo, and Stulz (2006).

4. Financially constrained firm may find it optimal to follow risk increasing strategies, in contrast with Froot, Scharfstein and Stein (1993).

5. Cash holdings should be used to cover operating losses rather than to finance investment, consistent with the evidence in the studies by Opler, Pinkowitz, Stulz and Williamson (1999) and Bates, Kahle, and Stulz (2009) and in the survey by Lins, Servaes, and Tufano (2010).

6. Firms should always increase their cash buffer when raising funds from outside investors, consistent with the evidence in Kim and Weisbach (2008) and McLean (2010).

7. Firms with better investment opportunities should find it optimal to accelerate investment with internal funds by decreasing the optimal level of cash holdings.

8. Firms with more tangible assets (with a higher liquidation value) should have lower cash holdings and should have a greater propensity to invest out of internal funds.

9. Time-varying capital supply should lead large, publicly traded firms to carry a significant cash buffer as a hedge against possible changes in the supply of capital.

While some of these predictions are shared with other models, many are novel and provide grounds for further empirical work on corporate policy choices.

Appendix

A. Proofs of the results in Section 3

To facilitate the proofs let us start by introducing some notation that will be of repeated use throughout the appendix. Let $\mathcal{L}_i$ denote the differential operator defined by

$$\mathcal{L}_i\phi(c) := \phi'(c)(rc+\mu_i) + \frac{\sigma^2}{2}\phi''(c) - \rho\phi(c),$$

set

$$\mathcal{F}\phi(c) := \max_{f\geq 0} \lambda\{\phi(c+f) - \phi(c) - f\}, \tag{38}$$

and denote by Θ the set of dividend and financing strategies such that

$$E_c\left[\int_0^{\tau_0} e^{-\rho s}(dD_s + f_{s-}dN_s)\right] < \infty$$

for all $c \geq 0$ where τ_0 is the first time that the firm's cash holdings fall to zero and $E_c[\cdot]$ denotes an expectation conditional on the initial value $C_{0-} = c$.

Let $\hat{V}_i(c)$ denote the value of the firm in the absence of growth option when the mean cash flow rate is equal to μ_i. In accordance with standard singular stochastic control results (see e.g. Fleming and Soner (1993, Chapter VIII)) we have that the Hamilton-Jacobi-Bellmann (HJB) equation is given by

$$\max\{\mathcal{L}_i\phi(c) + \mathcal{F}\phi(c), 1 - \phi'(c), \ell_i(c) - \phi(c)\} = 0 \tag{39}$$

where $\ell_i(c) := c + \varphi\mu_i/\rho$ denotes the liquidation value of the firm. Our first result shows that any classical solution to the HJB equation dominates the value of the firm.

Lemma 8 *If ϕ is a twice continuously differentiable solution to (39) then $\phi(c) \geq \hat{V}_i(c)$.*

Proof. Let ϕ be as in the statement, fix a strategy $(D, f) \in \Theta$ and consider the process

$$Y_t := e^{-\rho t}\phi(C_t) + \int_{0+}^{t} e^{-\rho s}(dD_s - f_{s-}dN_s)\,.$$

Using the assumption of the statement in conjunction with Itô's formula for semimartingales (see Dellacherie and Meyer (1980, Theorem VIII–25)) we get that $dY_t = dM_t - e^{-\rho t}dA_t$ where the process M is a local martingale and

$$\begin{aligned} dA_t = {} & (\phi(C_{t-} + f_{t-}) - \phi(C_{t-}) - f_{t-} - \mathcal{F}\phi(C_{t-}))dt \\ & + (\Delta D_t - \phi(C_{t-} - \Delta D_t) + \phi(C_{t-})) + (\phi'(C_{t-}) - 1)dD_t^c. \end{aligned}$$

The definition of $\mathcal{F}$ and the fact that $\phi' \geq 1$ then imply that A is nondecreasing and it follows that Y is a local supermartingale. The liquidation value being nonnegative we have

$$Z_t := Y_{t\wedge\tau_0} \geq -\int_0^{\tau_0} e^{-\rho s} f_{s-}dN_s$$

and since the random variable on the right hand side is integrable by definition of the set Θ we conclude that Z is a supermartingale. In particular,

$$\begin{aligned}
\phi(C_{0-}) &= \phi(C_0) - \Delta\phi(C_0) = Z_0 - \Delta\phi(C_0) \geq E_c[Z_{\tau_0}] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau_0}\phi(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau_0}\phi(C_{\tau_0}) + \int_{0}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_{0}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&\geq E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_{0}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] \qquad (40)
\end{aligned}$$

where the first inequality follows from the optional sampling theorem for supermartingales, the fifth equality follows from $C_{\tau_0} = 0$, and the last inequality follows from

$$\Delta D_0 + \Delta\phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-}-\Delta D_0}^{C_{0-}} (1 - \phi'(c))dc \leq 0$$

The desired result now follows by taking supremum over $(D, f) \in \Theta$ on both sides of (40). ■

Lemma 9 *Let $X \geq 0$ be fixed. The unique twice continuously differentiable solution to*

$$\begin{aligned}
\mathcal{L}_i\phi(c) - \lambda(\phi(X) - X + c - \phi(c)) = 0, \qquad & c \leq X, \qquad (41) \\
\phi(c) - \phi(X) + X - c = 0, \qquad & c \geq X,
\end{aligned}$$

is explicitly given by $\phi_i(c) = V_i(c \wedge X; X) + (c - X)^+$ where

$$V_i(c; X) \equiv \alpha_i(X)F_i(c) - \beta_i(X)G_i(c) + \frac{\lambda}{\lambda + \rho}\left(\frac{(r - \rho)X + \mu_i}{\rho} + \frac{rc + \mu_i}{\lambda + \rho - r}\right)$$

and the functions α_i, β_i are defined as in Proposition 1.

Lemma 10 *The general solution to the homogenous equation*

$$\lambda\phi_i(c) = \mathcal{L}_i\phi_i(c) \qquad (42)$$

is explicitly given by

$$\phi_i(c) = \gamma_1 F_i(c) + \gamma_2 G_i(c)$$

for some constants γ_1, γ_2 where the functions F_i, G_i are defined as in (7), (8).

Proof. The change of variable $\phi_i(c) = g_i(-(rc + \mu_i)^2/(r\sigma^2))$ transforms equation (42) for ϕ_i into Kummer's ODE for g_i and the conclusion now follows from standard results regarding this second order ODE. ■

Lemma 11 *The functions F_i and G_i satisfy*

$$F_i'(c)G_i(c) - F_i(c)G_i'(c) = -\frac{e^{-(rc+\mu_i)^2/(\sigma^2 r)}}{\sigma\, r^{-1/2}}$$

In particular, the ratio F_i/G_i is monotone decreasing.

Proof. The first claim follows from Abel's identity (see Hartman (1982, Section XI.2)). The second one follows because $(F_i/G_i)' = (F_i'G_i - F_iG_i')/G_i^2$. ■

Proof of Lemma 9. By application of Lemma 10 we have that the general solution to the second order ODE (41) is explicitly given by

$$V_i(c;X) = a_1F_i(c) + a_2G_i(c) + \frac{\lambda}{\lambda+\rho}\left(\phi_i(X) - X + \frac{(\rho+\lambda)c + \mu_i}{\lambda+\rho-r}\right)$$

for some constants $(a_1, a_2, \phi_i(X))$ and the proof will be complete once we show that these three unknowns are uniquely determined by the requirement that the solution be twice continuously differentiable. Using Lemma 11 in conjunction with the fact that F_i and G_i solve (42) we obtain that

$$\begin{aligned} -(F_i'(c)G_i''(c) - F_i''(c)G_i'(c)) &= 2\sigma^{-3}\sqrt{r}(\lambda+\rho)e^{-(rc+\mu_i)^2/(\sigma^2 r)}, \\ F_i''(c)G_i(c) - F_i(c)G_i''(c) &= 2\sigma^{-3}\sqrt{r}(rc+\mu_i)e^{-(rc+\mu_i)^2/(\sigma^2 r)}. \end{aligned} \tag{43}$$

Combining these identities with the smooth pasting and high contact conditions $V_i'(X;X) = 1$, $V_i''(X;X) = 0$ then gives $a_1 = \alpha_i(X)$, $a_2 = -\beta_i(X)$ and it now follows from (43) that

$$a_1F_i(X) + a_2G_i(X) = \frac{(\rho-r)(rX+\mu_i)}{(\lambda+\rho-r)(\lambda+\rho)}.$$

Substituting this identity into the value matching condition

$$\phi_i(X) = V_i(X;X) = a_1F_i(X) + a_2G_i(X) + \frac{\lambda}{\lambda+\rho}\left(\phi_i(X) + \frac{rX+\mu_i}{\lambda+\rho-r}\right)$$

and solving the resulting equation gives $V_i(X;X) = (rX+\mu_i)/\rho$ and completes the proof. ■

Lemma 12 *The function $V_i(c;X)$ is increasing and concave with respect to $c \leq X$, and strictly monotone decreasing with respect to X.*

In order to prove Lemma 12 we will rely on the following three useful results.

Lemma 13 *Suppose that k is a solution to*

$$\mathcal{L}_i k(c) + \phi(c) = 0 \tag{44}$$

for some ϕ. Then, k does not have negative local minima if $\phi(c) \geq 0$ and k does not have positive local maxima if $\phi(c) \leq 0$.

Proof. At a local minimum we have $k'(c) = 0$, $k''(c) \geq 0$ and the claim follows from (44) and the nonnegativity of ϕ. The case of a non-positive ϕ is analogous. ■

Lemma 14 *Suppose that k is a solution to (44) for some $\phi(c) \leq 0$ and that $k'(c_0) \leq 0$, $k(c_0) \geq 0$ and $|k(c_0)| + |k'(c_0)| + |\phi(c_0)| > 0$. Then, $k(c) > 0$ and $k'(c) < 0$ for all $c < c_0$.*

Proof. Suppose on the contrary that $k'(c)$ is not always negative for $c < c_0$ and let z be the largest value of $c < c_0$ at which $k'(c)$ changes sign. Then, z is a positive local maximum and the claim follows from Lemma 13. ■

Lemma 15 *Suppose that k is a solution to (44) for some ϕ such that $\phi'(c) \leq 0$ and that $k'(c_0) \geq 0$, $k''(c_0) \leq 0$ and $|k'(c_0)| + |k''(c_0)| + |\phi'(c_0)| > 0$. Then, $k'(c) > 0$ and $k''(c) < 0$ for all $c < c_0$ and $k''(c) > 0$ for $c > c_0$. In particular,*

$$k'(c_0) \;=\; \min_{c \geq 0} k'(c)\,.$$

Proof. Differentiating (44) shows that $m = k'$ is a solution to $\mathcal{L}_i m(c) + rm(c) + \phi'(c) = 0$ and the conclusion follows from Lemma 14. The case $c > c_0$ is analogous. ■

Proof of Lemma 12. As is easily seen the function

$$k(c) = V_i(c, X) - \frac{\lambda}{\lambda + \rho}\left(V_i(X; X) - X + \frac{(\rho + \lambda)c + \mu_i}{\lambda + \rho - r}\right)$$

is a solution to (42) and satisfies $k'(X) = 1 > 0$ as well as $k''(X) = 0$. In conjunction with Lemma 15 this implies that $k(c)$, and hence also $V_i(c; X)$, is increasing and concave for $c \leq X$.

To establish the required monotonicity, let $X_1 < X_2$ be fixed and consider the function $m(c) = V_i(c; X_1) - V_i(c; X_2)$. Using the first part of the proof it is easily seen that m solves

$$\mathcal{L}_i m(c) - \lambda m(c) - \lambda(1 - r/\rho)(X_1 - X_2) = 0$$

with the boundary conditions $m'(X_1) = 1 - V_i'(X_1; X_2) < 0$, $m''(X_1) = -V_i''(X_1; X_2) \geq 0$. Thus it follows by a straightforward modification of Lemma 15 that m is monotone decreasing and it only remains to show that $m(X_1) > 0$. To this end, observe that

$$\begin{aligned} m(X_1) &= V_i(X_1; X_1) - V_i(X_1; X_2) \\ &= V_i(X_1; X_1) - V_i(X_2; X_2) + \int_{X_2}^{X_1} V_i'(c; X_2) dc \\ &\geq V_i(X_1; X_1) - V_i(X_2; X_2) + X_2 - X_1 = (r/\rho - 1)(X_1 - X_2) > 0 \end{aligned}$$

where the first inequality follows from $V_i'(X; X) = 1$ and the first part of the proof, and the last inequality follows from the fact that by assumption $\rho > r$. ■

Lemma 16 *The unique solution to the free boundary problem (1)–(6) is given by*

$$V_i(c) = V_i(c \wedge C_i^*; C_i^*) + (c - C_i^*)^+$$

where C_i^* is the unique solution to $V_i(0, X) = \ell_i(0)$. The function V_i is a twice continuously differentiable solution to (39).

Proof. By Lemma 9 we have that $V_i(c)$ is twice continuously differentiable, satisfies (1) and solves (2) subject to (5), (4) and (6) so we only need to show that (3), or equivalently

$$V_i(0; C_i^*) = \ell_i(0) \tag{45}$$

uniquely determines the value of C_i^*. By Lemma 12 we have that $V_i(0; X)$ is monotone decreasing. On the other hand, a direct calculation shows that $V_i(0; 0) = \mu_i/\rho > 0$, $V_i(0; \infty) < 0$ and it follows that (45) has a unique solution. To complete the proof it remains to show that V_i is a solution to the HJB equation. Using the concavity of $V_i(c)$ in conjunction with the smooth pasting condition we obtain that $1 - V_i'(c)$ is negative below the threshold C_i^* and zero otherwise so that

$$\ell_i(c) - V_i(c) = \int_0^c (1 - V_i'(x))dx \leq 0.$$

On the other hand, using the concavity of $V_i(c) = V_i(c; C_i^*)$ in conjunction with Lemma 9 and the smooth pasting condition we obtain

$$\begin{aligned}(\mathcal{L}_i + \mathcal{F})V_i(c) &= \mathcal{L}_i V_i(c) + 1_{\{c<C_i^*\}}(V_i(C_i^*) - X + c - V_i(c)) = 1_{\{c\geq C_i^*\}}\mathcal{L}_i V_i(c) \\ &= (r-\rho)(c - C_i^*)^+ \leq 0\end{aligned}$$

Combining the above results shows that V_i is a solution to (39) and completes the proof. ■

Proof of Proposition 1. Combining the results of Lemmas 8 and 16 shows that $V_i \geq \hat{V}_i$. In order to establish the reverse inequality, consider the dividend and financing strategy defined by $D_t^* = L_t$ and $f_t^* = (C_i^* - C_t)^+$ where the process C evolves according to

$$dC_t = (rC_{t-} + \mu_i)dt + \sigma dB_t - dD_t^* + f_{t-}^* dN_t$$

with initial condition $C_{0-} = c \geq 0$ and $L_t = \sup_{s\leq t}(X_t - C_i^*)^+$ where

$$dX_t = (rX_{t-} + \mu_i)dt + \sigma dB_t + (C_i^* - X_{t-})^+ dN_t\,.$$

In order to show that the strategy (D^*, f^*) is admissible we start by observing that due to standard properties of Poisson processes we have

$$E_c\left[\int_0^\infty e^{-\rho t} f_{t-}^* dN_t\right] \leq E_c\left[\int_0^\infty e^{-\rho t} C_i^* dN_t\right] = \frac{\lambda C_i^*}{\rho}$$

where the inequality follows from the definition of f^*. Using this bound in conjunction with Itô's lemma and the assumption that $r < \rho$ we then obtain that

$$\begin{aligned}E_c\left[\int_0^t e^{-\rho s} dD_s^*\right] &= C_0 + E_c\left[\int_0^t e^{-\rho s}((r-\rho)C_{s-} + \mu_i)ds + \int_0^t e^{-\rho s} f_{s-}^* dN_s\right] \\ &\leq C_0 + E_c\left[\int_0^\infty e^{-\rho s}\mu_i ds + \int_0^\infty e^{-\rho s} f_{s-}^* dN_s\right] \leq C_0 + \frac{1}{\rho}(\mu_i + \lambda C_i^*)\end{aligned}$$

holds for any finite t and it now follows from Fatou's lemma that $(D^*, f^*) \in \Theta$. Applying Itô's formula for semimartingales to the process

$$Y_t = e^{-\rho(t\wedge\tau_0)} V_i(C_{t\wedge\tau_0}) + \int_{0+}^{t\wedge\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)$$

and using the definition of (D^*, f^*) in conjunction with the fact that the function V_i solves the HJB equation we obtain that the process Y is a local martingale. Now, using the fact that $C_t \in [0, C_i^*]$ for all $t \geq 0$ together with the increase of V_i we deduce that

$$|Y_\theta| \leq |V_i(C_i^*)| + \int_0^\infty e^{-\rho t}(dD_t^* + f_{t-}^* dN_t)$$

for any stopping time θ and, since the random variable on the right hand side is integrable, we conclude that the process Y is a uniformly integrable martingale. In particular, we have

$$\begin{aligned} V_i(c) = Y_{0-} &= Y_0 - \Delta Y_0 = Y_0 + \Delta D_0^* = E_c[Y_{\tau_0}] + \Delta D_0^* \\ &= E_c\left[e^{-\rho\tau_0} V_i(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)\right] + \Delta D_0^* \\ &= E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)\right] \end{aligned}$$

where the third equality follows from the definition of V_i and the fourth one follows from the martingale property of Y. This shows that $V_i \geq \hat{V}_i$ and establishes the optimality of (D^*, f^*). ■

Lemma 17 *The level of cash holdings C_i^* that is optimal for a firm with no growth option is monotone decreasing in λ and φ.*

Proof. Monotonicity in φ follows from the definition of C_i^* and the monotonicity of ℓ_i. To establish the required monotonicity in λ it suffices to show that $V_i(0; X, \lambda)$ is monotone decreasing in λ. Indeed, in this case we have

$$\ell_i(0) = V_i(0; C_i^*(\lambda_1), \lambda_1) \leq V_i(0; C_i^*(\lambda_1); \lambda_2)$$

for all $\lambda_1 < \lambda_2$ and therefore $C_i^*(\lambda_2) \leq C_i^*(\lambda_1)$ due to the fact that $V_i(0; X, \lambda)$ is decreasing in X. To establish the required monotonicity observe that $V_i(X; X, \lambda) = \frac{rX+\mu_i}{\rho}$ does not depend on λ. As a result, it follows from Lemma 9 that the function defined by

$$k(c) = V_i(c; X, \lambda_1) - V_i(c; X, \lambda_2)$$

for some $\lambda_1 < \lambda_2$ satisfies

$$k(X) = k'(X) = k''(X) = k^{(3)}(X) = k^{(4)}(X) = 0$$

and solves the ODE

$$\mathcal{L}_i k(c) - \lambda_1 k(c) = (\lambda_2 - \lambda_1)(V_i(X; X, \lambda_2) - V_i(c; X, \lambda_2) - (X - c)). \tag{46}$$

Since, by Lemma 12, $V_i(c; X, \lambda_2)$ is concave in c and $V_i'(X; X, \lambda_2) = 1$, the right hand side of (46) is nonnegative for all $c \leq X$. It follows by a slight modification of Lemma 13 that $k(c)$ cannot have a positive local maximum. Since

$$k^{(5)}(X) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2)V_i^{(3)}(X; X, \lambda_2) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2)(\rho - r) < 0\,,$$

we conclude that k is decreasing in a small neighborhood of X. Therefore, $k(c)$ is decreasing for all $c \leq X$ and hence $k(c) > k(X) = 0$ for all $c \leq X$. ■

B. Proof of Proposition 2

The proof of Proposition 2 will be based on a series of lemmas. To facilitate the presentation of the results, let $\hat{V}$ denote the value of the firm and Π denote the set of triples $\pi = (\tau, D, f)$ where τ is a stopping time that represents the firm's investment time and $(D, f) \in \Theta$ is an admissible dividend and financing strategy.

Lemma 18 *The value of the firm satisfies*

$$\hat{V}(c) = \sup_{\pi \in \Pi} E_c \left[\int_0^{\tau \wedge \tau_0} e^{-\rho t} \left(dD_t - f_{t-} dN_t \right) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_1(C_\tau) \right].$$

In particular, if $V_0(c) \geq V_1(c - K)$ for all $c \geq K$ then it is optimal to abandon the growth option.

Proof. The proof of the first part follows from standard dynamic programming arguments and therefore is omitted. To establish the second part assume that $V_0(c) \geq V_1(c - K)$ for all $c \geq K$ and observe that $\Delta C_\tau = -K + 1_{\{\tau \in \mathcal{N}\}} f_{\tau-}$ where $\mathcal{N}$ denotes the set of jump times of the Poisson process. Using this identity in conjunction with the first part we obtain

$$V(c) \leq \sup_{\pi \in \Pi} E_c \Bigg[\int_0^{\tau \wedge \tau_0} e^{-\rho t} (dD_t - f_{t-} dN_t) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_0(C_{\tau-} + 1_{\{\tau \in \mathcal{N}\}} f_{\tau-}) \Bigg].$$

and the desired result follows since the right hand side of this inequality is equal to $V_0(c)$ by standard dynamic programming arguments. ■

In order to establish Proposition 2 it now suffices to show that $V_0(c) \geq V_1(c - K)$ for all $c \geq K$ if and only if $K \geq K^*$. This is the objective of the following:

Lemma 19 *The constant K^* is nonnegative and the following are equivalent:*

(1) $K \geq K^*$,

(2) $K \geq V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*)$,

(3) $V_0(c) \geq V_1(c - K)$ *for all* $c \geq K$.

Proof. The equivalence of (1) and (2) follows from the definition of K^* and the fact that by Proposition 1 we have

$$V_i(C_i^*) = \frac{r C_i^* + \mu_i}{\rho}.$$

In order to show that the constant K^* is nonnegative we argue as follows: Since $\mu_0 < \mu_1$, the set of feasible strategies for V_0 is included in the set of feasible strategies for V_1. It follows that $V_0 \leq V_1$ and combining this with the definition of C_i^* shows that

$$K^* = V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*) = \max_{C \geq 0} \{V_1(C) - C\} - \max_{C \geq 0} \{V_0(C) - C\} \geq 0.$$

To establish the implication (1) $\Rightarrow$ (3) it suffices to show that under (1) we have $V_1(c-K^*) \leq V_0(c)$ for all $c \geq K^*$. Indeed, if that is the case then (3) also holds since

$$V_1(c-K) \leq V_1(c-K^*), \qquad c \geq K \geq K^*$$

due to the increase of the function V_1. For $c \geq K^* \vee C_0^*$ the concavity of the function V_1 and the fact that the function V_0 is linear with slope one above the level C_0^* imply that

$$\begin{aligned} V_1(c-K^*) &\leq V_1(C_1^*) + (c - K^* - C_1^*) \\ &= V_0(c) + C_0^* - V_0(C_0^*) - K^* - C_1^* = V_0(c) \end{aligned}$$

so it remains to prove the result for $c \in [K^*, C_0^*]$. Consider the function $k(c) = V_0(c) - V_1(c-K^*)$. Using Lemma 9 in conjunction with the fact that $C_0^* < C_1^* + K^*$ by Lemma 20 below we have that the function k is a solution to

$$\mathcal{L}_0 k(c) - \lambda k(c) + (-\mu_1 + \mu_0 + rK^*)V_1'(c-K^*) = 0$$

on the interval $[K^*, C_0^*]$. Combining Lemma 20 below with the increase of V_1 shows that the last term on the left hand side of this equation is positive and since

$$\begin{aligned} k(C_0^*) &= V_0(C_0^*) - V_1(C_0^* - K^*) \geq V_0(C_0^*) - V_1(C^*) - (C_0^* - K^* - C_1^*) = 0, \\ k'(C_0^*) &= V_0'(C_0^*) - V_1'(C_0^* - K^*) = 1 - V_1'(C_0^* - K^*) \geq 0 \end{aligned}$$

by the concavity of V_1, we can apply Lemma 14 to conclude that $k(c) \geq 0$ for all $c \leq C_0^*$. Finally, the implication (3) $\Rightarrow$ (2) follows by taking $c > C_0^* \vee (C_1^* + K)$ ■

Lemma 20 *We have* $C_0^* < C_1^* + K^*$ *and* $\mu_1 - \mu_0 - rK^* > 0$.

Proof. The definition of the constant K^* implies that the first inequality is equivalent to the second which is in turn equivalent to

$$C_1^* - C_0^* > \frac{\mu_0 - \mu_1}{r}.$$

To prove this inequality, it suffices to show that in the absence of a growth option the optimal level of cash holdings $C_i^* = C^*(\mu_i)$ satisfies

$$-\frac{\partial C^*(\mu)}{\partial \mu} < \frac{1}{r}. \tag{47}$$

By an application of Lemma 16, we have that

$$\tilde{V}(0; C^*(\mu), \mu) + \frac{\lambda}{\lambda + \rho}\left(\frac{\mu + (r-\rho)C^*(\mu)}{\rho} + \frac{\mu}{\lambda + \rho - r}\right) = \frac{\varphi \mu}{\rho}$$

where the function $\tilde{V}$ is defined by

$$\tilde{V}(0; X, \mu) = \alpha(X; \mu)F(0; \mu) - \beta(X; \mu)G(0; \mu) \tag{48}$$

for α and β as in equations (9), (10) albeit with $\mu_i = \mu$. Using (7), (8) in conjunction with the definition of the functions α and β we obtain

$$\tilde{V}_\mu(0; X, \mu) = \frac{1}{r}\left(\tilde{V}_c(0; X, \mu) + \tilde{V}_X(0; X, \mu)\right).$$

where a subscript denotes a partial derivative and it follows that

$$\frac{\partial C^*(\mu)}{\partial \mu} = \frac{-\tilde{V}_X(0; C^*(\mu), \mu)/r - \tilde{V}_c(0; C^*(\mu); \mu)/r + \varphi/\rho - B}{\tilde{V}_X(0; C^*(\mu), \mu) - A}$$

where we have set

$$A = \frac{\lambda}{\lambda + \rho}\left(1 - \frac{r}{\rho}\right), \quad B = \frac{\lambda}{\lambda + \rho}\left(\frac{1}{\rho} + \frac{1}{\lambda + \rho - r}\right).$$

By Lemma 12 we have that the function $\tilde{V}$ is decreasing in X and since $A > 0$ it follows that the validity of equation (47) is equivalent to

$$-\tilde{V}_X(0; C^*(\mu), \mu) - \tilde{V}_c(0; X; \mu) - r\left(B - \frac{\varphi}{\rho}\right) < A - \tilde{V}_X(0; C^*(\mu), \mu),$$

which in turn follows from

$$\tilde{V}_c(0; X; \mu) + r\left(B - \frac{1}{\rho}\right) > 0. \tag{49}$$

Since the difference $\tilde{V} - V$ is a linear function of c we have from Lemma 12 that the function $\tilde{V}$ is concave. Thus, it follows from the smooth pasting condition that

$$\tilde{V}_c(0; C^*(\mu), \mu) = V_c(0; C^*(\mu), \mu) - \frac{\lambda}{\lambda + \rho - r} \geq V_c(C^*(\mu); C^*(\mu), \mu) - \frac{\lambda}{\lambda + \rho - r} = \frac{\rho - r}{\lambda + \rho - r}$$

and combining this inequality with a straightforward calculation shows that (49) holds. ■

C. Additional results

The proofs of the additional results in the paper are constructed along the same lines as the proof of Proposition 1. These proofs are relegated to the Supplementary Appendix.

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Figure 1: Cash holdings for a firm with no growth option

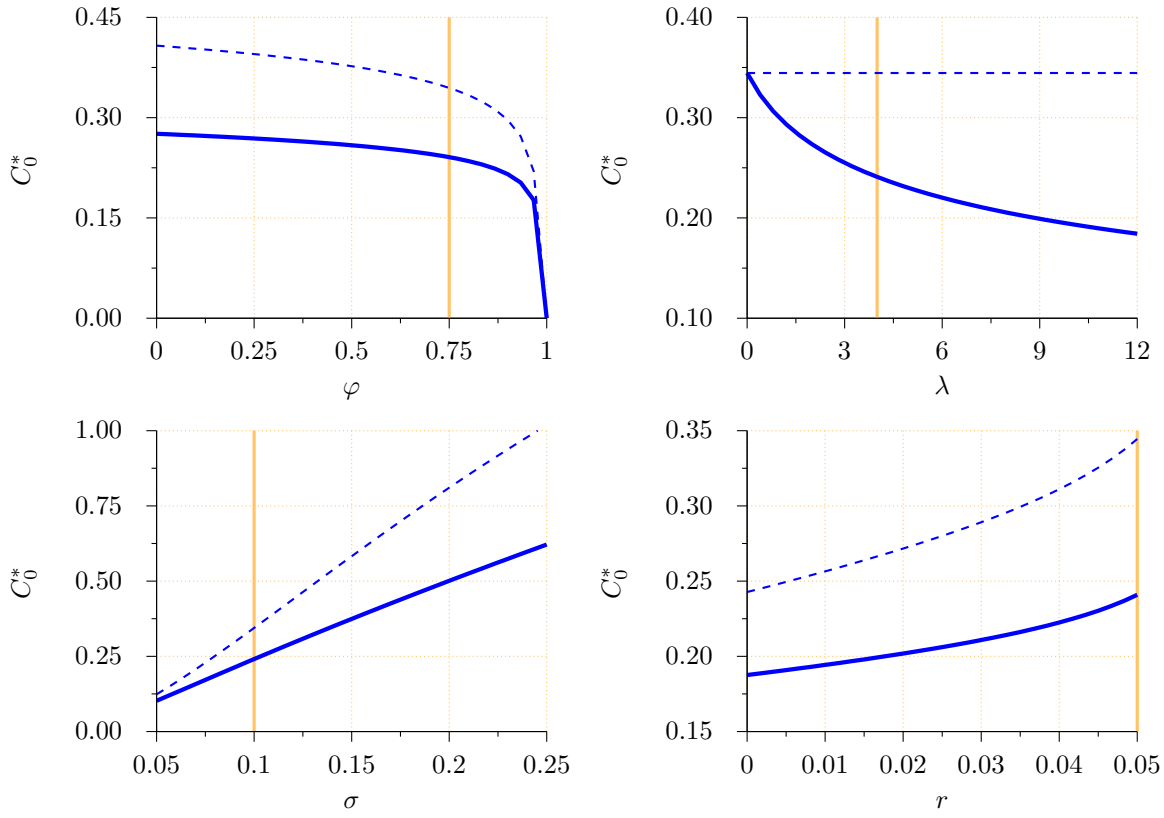

Figure 1 plots the value-maximizing cash buffer C_0^* when the firm has no access to external funds (dashed line) and when the firm has access to external funds (solid line) as a function of the recovery rate on assets φ, the arrival rate of investors λ, the reinvestment rate r, and the cash flow volatility σ. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, and $\mu_0 = .1$. In each panel the vertical line indicates the base value of the parameter.

Figure 2A: Cash holdings for a firm with a growth option

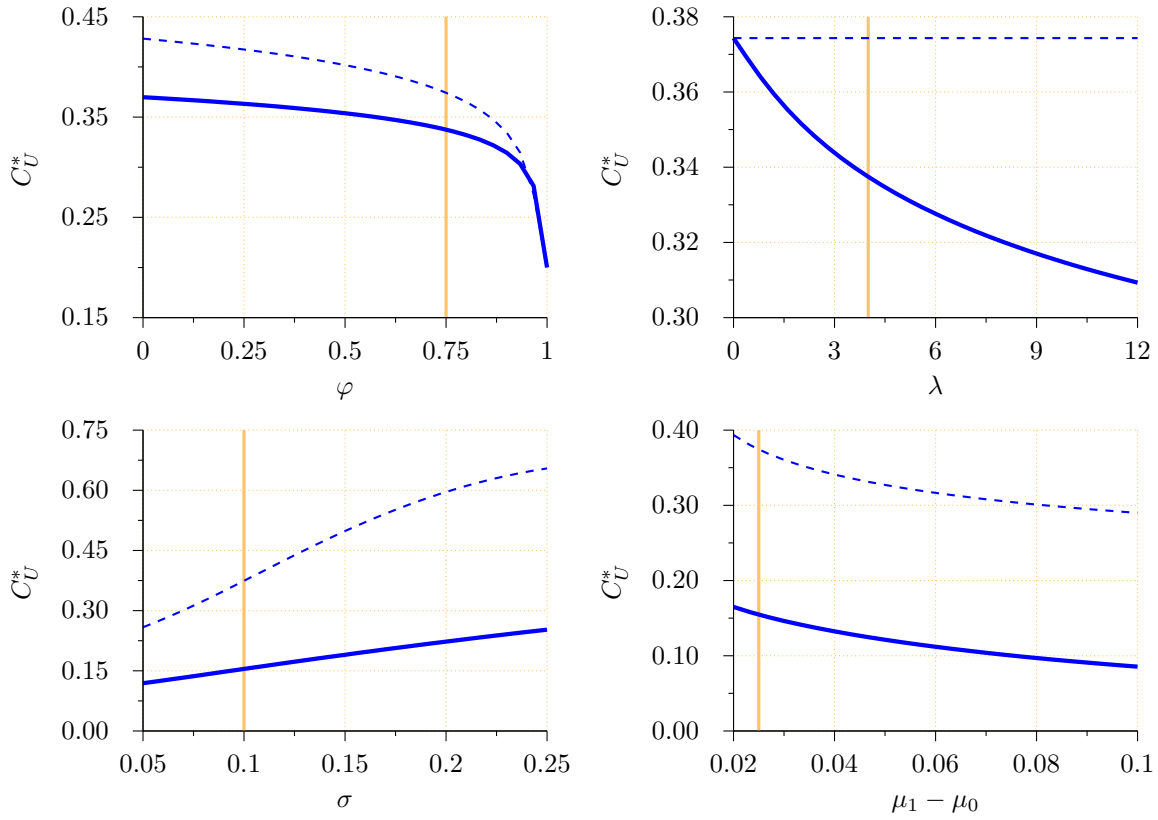

Figure 2A plots the value maximizing cash buffer C_U^* when the firm has no access to external funds (dashed line) and when the firm has access to external funds (solid line) as a function of the recovery rate on assets φ, the arrival rate of investors λ, the cash flow volatility σ, and the growth potential of the firm $\mu_1 - \mu_0$. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, $\mu_0 = .1$, $\mu_1 = 0.125$, and $K = .2$. In each panel the vertical line indicates the base value of the parameter.

Figure 2B: Change in the optimal cash buffer

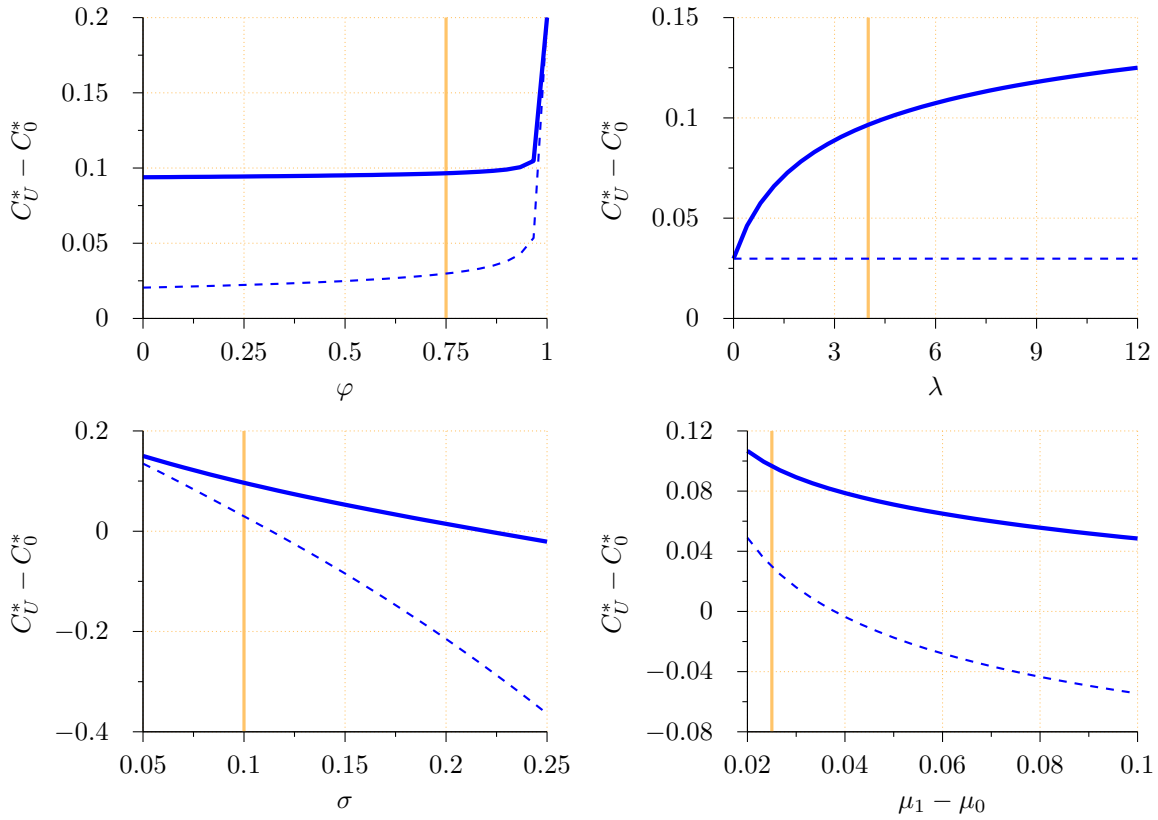

Figure 2B plots the change in the value maximizing cash buffer $C_U^* - C_0^*$ that is due to the presence of a growth option when the firm has no access to external funds (dashed line) and when the firm has access to external funds (solid line) as a function of the recovery rate on assets φ, the arrival rate of investors λ, cash flow volatility σ, and the growth potential of the firm $\mu_1 - \mu_0$. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, $\mu_0 = .1$, $\mu_1 = 0.125$, and $K = .2$. In each panel the vertical line indicates the base value of the parameter.

Figure 3: Unconditional probability of investment

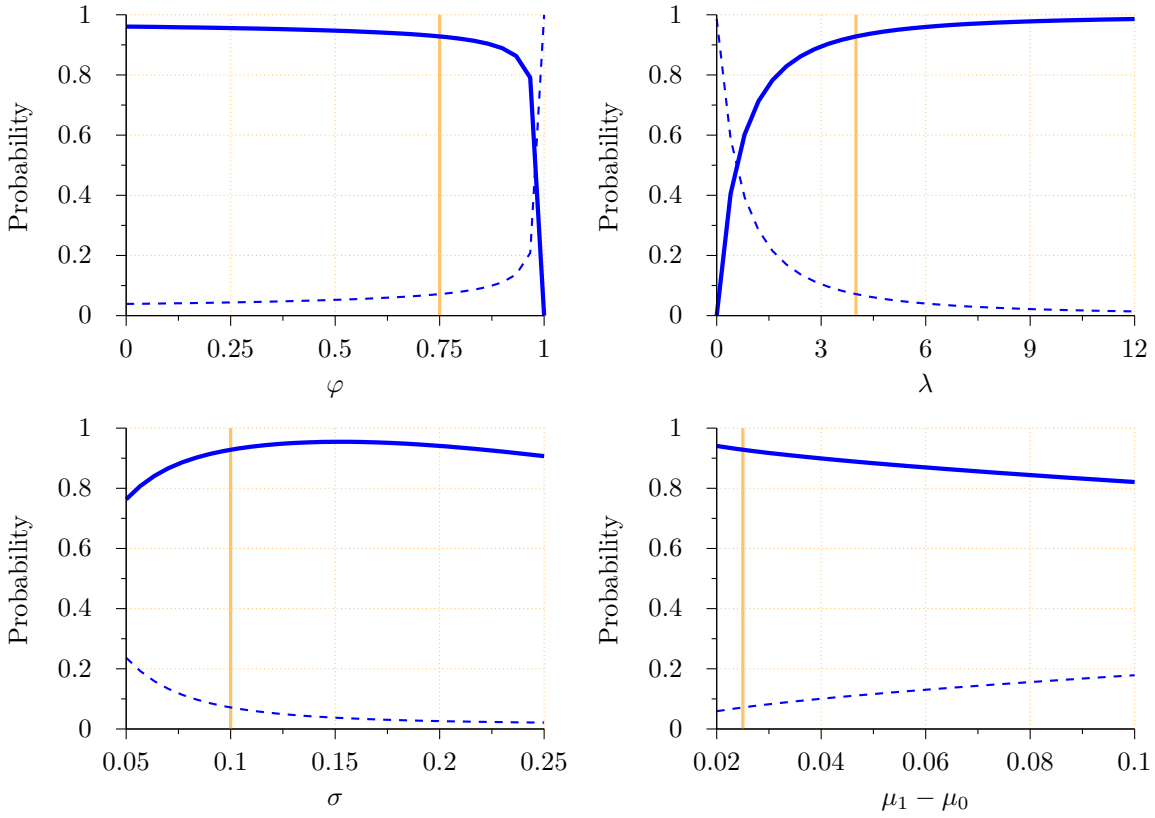

Figure 3 plots the probability of investment using internal funds (dashed line) and investment using external funds (solid line) given that the firm's cash buffer equals the investment cost as functions of the recovery rate on assets φ, the arrival rate of investors λ, the reinvestment rate r, the cash flow volatility σ, and the growth potential of the firm $\mu_1 - \mu_0$. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, $\mu_0 = .1$, $\mu_1 = 0.125$, and $c_0 = K = .2$. In each panel the vertical line indicates the base value of the parameter.

Figure 4: Capital supply, investment and default

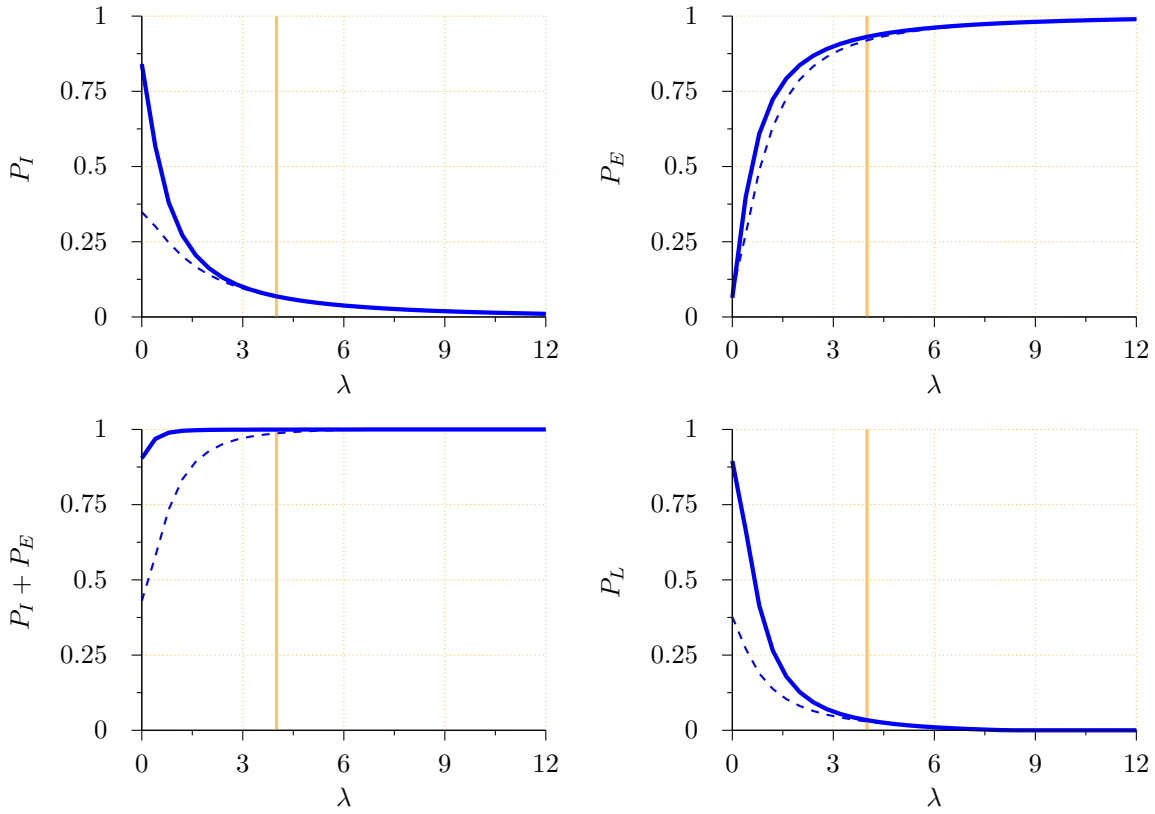

Figure 4 plots the probability of investment using internal funds, the probability of investment using external funds, and the probability of liquidation over a 1-year (dashed line) and 3-year (solid line) horizon given that the firm's cash buffer equals the investment cost as functions of the arrival rate of investors λ. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, $\mu_0 = .1$, $\mu_1 = 0.125$, and $c_0 = K = .2$. In each panel the vertical line indicates the base value of the parameter.

Figure 5: Cash holdings with financing costs

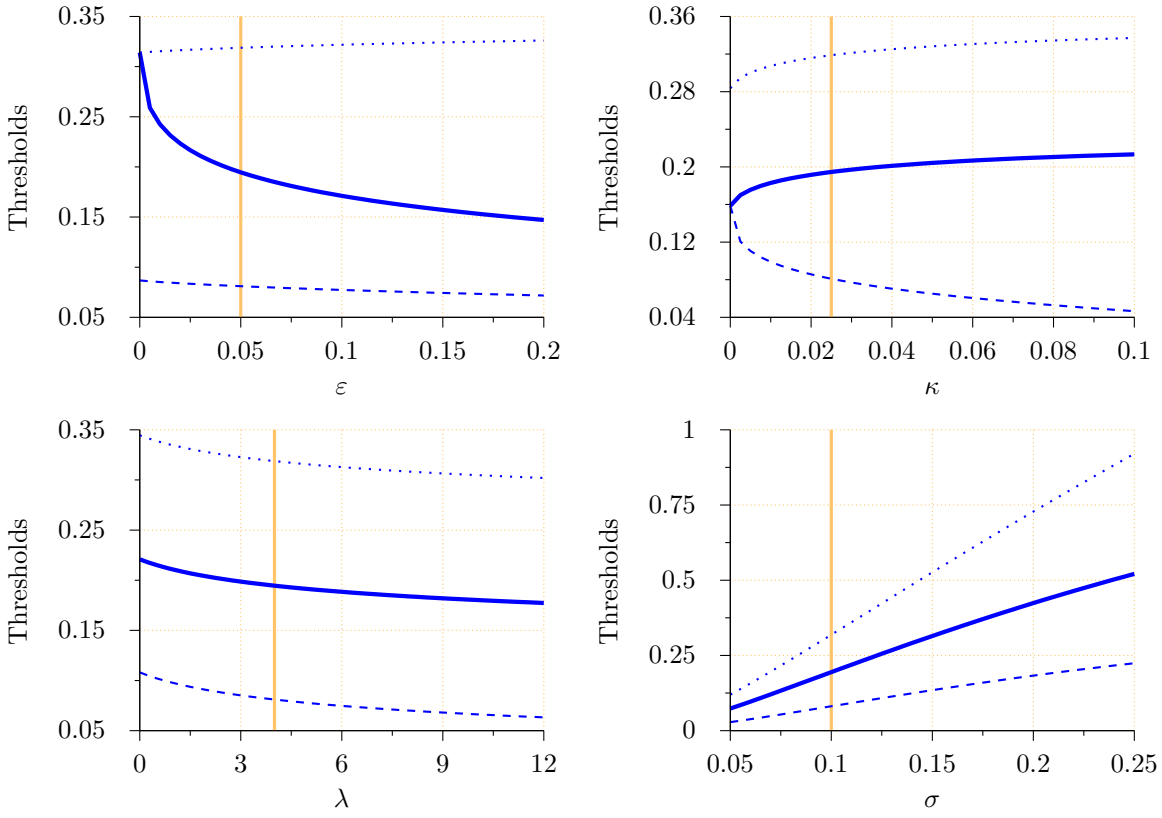

Figure 5 plots the value maximizing cash thresholds $\underline{C}^*$ (dashed line), $\overline{C}^*$ (solid line) and C^* (dotted line) as functions of the variable cost of external financing ε, the fixed cost of external financing κ and the arrival rate of investors λ. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, $\mu_0 = .1$, $\varepsilon = .05$, and $\kappa = .025$. In each panel the vertical line indicates the base value of the parameter.

Figure 6: Cash holdings with time varying capital supply

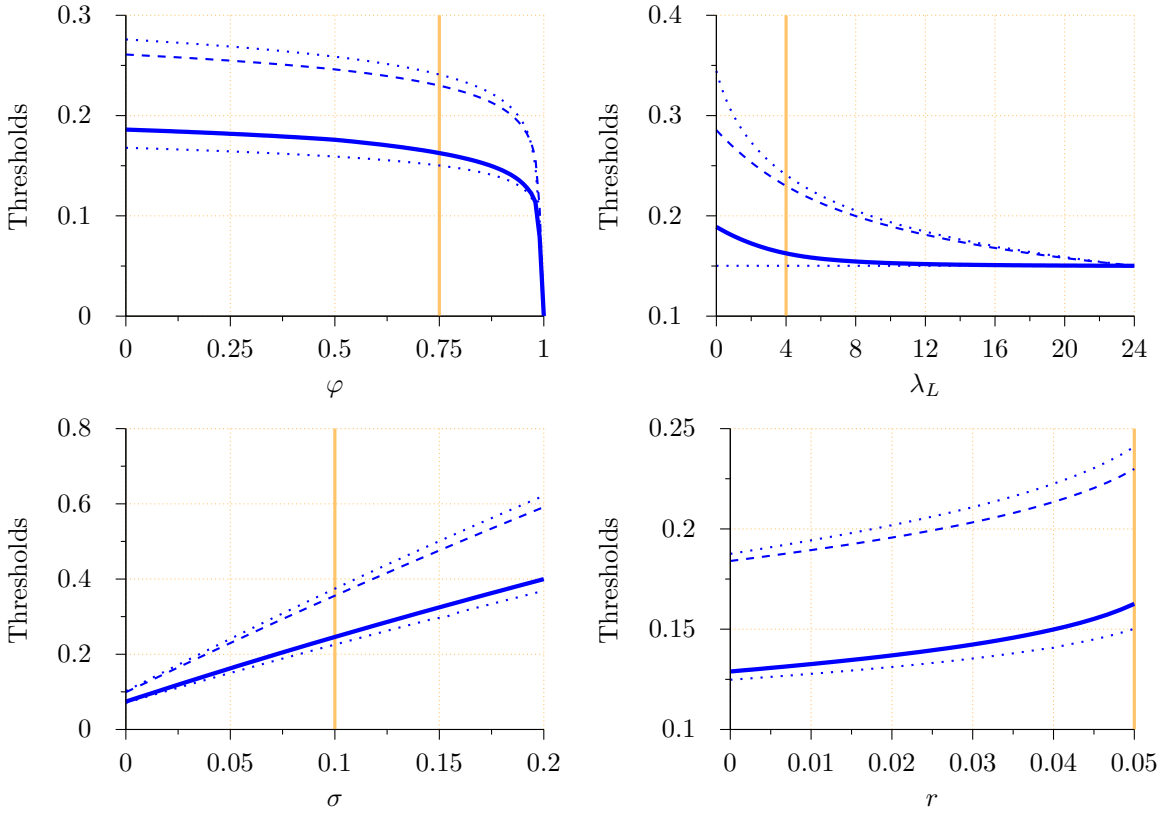

Figure 6 plots the value maximizing cash holdings C_H^* (solid line) and C_L^* (dashed line) as functions of the recovery rate on assets φ, the arrival rate of investors in the low state λ_L, the volatility of cash flows σ and the interest rate. The base parametrization is $\rho = .06$, $r = .05$, $\varphi = .75$, $\sigma = .1$, $\mu_0 = .1$, $\lambda_L = 4$, $\lambda_H = 24$, $\pi_{LH} = 1.154$, and $\pi_{HL} = 0.233$. In each panel the vertical line indicates the base value of the parameter.

Supplementary appendix

A. Proof of Proposition 3 and some additional results.

Denote the value of the firm by $\hat{V}$. Proposition 3 directly follows from the following:

Lemma 21 *The unique smooth solution to the free boundary problem defined by (12), (13), (14) and (15) is given by (W, C_W^*) where the function W is defined by (19) and the constant C_W^* is the unique solution to (22). The function W is increasing, concave and satisfies $W(c) \leq \hat{V}(c)$.*

Proof. The results follow by arguments similar to those we used in the proof of Proposition 1. We omit the details. ■

The following result follows by direct calculation.

Lemma 22 *The unique solution $\psi(c;K)$ to equation (12) satisfying $\psi(0;K) = \ell_0$ and $\psi(K;K) = \ell_1$ is explicitly given by*

$$\psi(c;K) \;=\; a_1(K)\,F_0(c) \;-\; b_1(K)\,G_0(c) \;+\; \Phi(c,K)$$

where the function a_1 and b_1 are defined by

$$a_1(K) \;=\; \frac{G_0(K)(\ell_0 - \Phi(0;K)) - G_0(0)(\ell_1 - \Phi(K;K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}$$

and

$$b_1(K) \;=\; \frac{F_0(K)(\ell_0 - \Phi(0;K)) - F_0(0)(\ell_1 - \Phi(K;K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}$$

with $\Phi = \Phi(c;K)$ as in equation (21).

Lemma 23 *Let $\psi'(K;K) < V_1'(0)$. Then, the unique solution to the free boundary problem defined by (12), (13), (16), (17) is given by (U, C_U^*) where the function U is defined by (20) and the constant C_U^* is the unique solution to (22) with*

$$\xi_G(x) = e^{(rx+\mu_0)^2/(r\sigma^2)}\sigma r^{-1/2}\left(G_0'(x)v_1(x-K) - G_0(x)v_1'(x-K)\right),$$
$$\xi_F(x) = e^{(rx+\mu_0)^2/(r\sigma^2)}\sigma r^{-1/2}\left(F_0'(x)v_1(x-K) - F_0(x)v_1'(x-K)\right),$$

and

$$v_1(x) = V_1(x) - \Phi(x+K)\,. \tag{50}$$

If $\psi'(K;K) > V_1'(0)$ then we let $C_U^* = K$ and the function U is defined by (20) with $\xi_G(K) = a_1(K)$, $\xi_F(K) = b_1(K)$.

Proof. Using arguments similar to those of the proof of Lemma 9 it can be shown that the unique solution to (12) such that (16) and (17) hold for $c = C_U^*$ is increasing and given by

$$U(c) = \xi_G(C_U^*)F_0(c) - \xi_F(C_U^*)G_0(c) + \Phi(c)$$

for $c \leq C_U^*$. As a result, the first part of the proof will be complete once we show that the value matching condition at zero

$$U(0) = \xi_G(C_U^*)F_0(0) - \xi_F(C_U^*)G_0(0) + \Phi(0) = \ell_0 \tag{51}$$

admits a unique solution $C_U^* \leq C_1^* + K$ when $\psi'(K;K) \leq V_1'(0)$. To this end we start by observing that, as a result of Lemma 24 below, finding a solution to the free boundary problem (12), (13), (16), (17) is equivalent to finding a linear function ϕ that is tangent to the graph of the function $\hat{v}_1$ defined by

$$v_1(c-K) = F_0(c)\hat{v}_1(Z(c)) = F_0(c)\hat{v}_1\left(\frac{G_0(c)}{F_0(c)}\right) \tag{52}$$

and such that

$$\phi(Z(0))F_0(0) = \ell_0 - \Phi(0).$$

A direct calculation using the results of Lemmas 16 and 20 shows that

$$\mathcal{L}_0 v_1(c-K) - \lambda v_1(c) = (r-\rho)(c - C_1^* - K)^+ + (-\mu_1 + \mu_0 + rK)V_1'(c-K) \leq 0$$

for all $c \geq K$ and it now follows from Lemma 24 that $\hat{v}_1(y)$ is concave for all $y \geq Z(K)$. On the other hand, since V_1 is concave, we obtain

$$v_1'(c-K) = V_1(c-K) - \frac{\lambda}{\lambda+\rho-r} \geq V_1'(C_1^*) - \frac{\lambda}{\lambda+\rho-r} = \frac{\rho-r}{\lambda+\rho-r} > 0$$

and it follows $v_1(c-K)$ is positive for sufficiently large values of c. Since F_0 is nonnegative and decreasing, this implies that the ratio $v_1(c-K)/F_0(c)$ is increasing for large c and it now follows from Lemma 24 below that $\hat{v}_1(y)$ is increasing for large values of y and is therefore globally increasing in $y \geq Z(K)$ since it is concave in that region.

Since by assumption $\psi'(K;K) \leq V_1'(0)$ we have that the line passing through the points

$$(Z(0), (\ell_0 - \Phi(0))/F_0(0))$$

and

$$(Z(K), (\ell_1 - \Phi(0))/F_0(0))$$

has a higher slope at $y = Z(K)$ than $\hat{v}_1$. Using the concavity and increase of $\hat{v}_1$, it is then immediate that there exists a unique line passing through $(Z(0), (\ell_0 - \Phi(0))/F_0(0))$ that is tangent to $\hat{v}_1$ at some $y^* > Z(K)$. Setting $C_U^* = Z^{-1}(y^*)$ proves the existence of a unique solution to the value matching condition (51). Since $\hat{v}_1$ is concave, it lies below its tangent line at y^* and, transforming back to $V_1(c-K)$ and $U(c)$, we get $U(c) \geq V_1(c-K)$.

In order to prove that $U(c) \leq \hat{V}(c)$, and thus complete the proof, consider the investment, dividend and financing strategy π^U defined by $\tau = \tau_N \wedge \tau_U^*$, $D^U = 0$ and

$$f_t^U = (C_1^* + K - C_{t-})^+ \tag{53}$$

where τ_N denotes the first jump time of the Poisson process and τ_U^* denotes the first time that the firm's cash reserves reach the level C_U^*. As is easily seen, we have

$$E_c\left[\int_0^{\tau_0} e^{-\rho t}\left(dD_t^U + f_{t-}^U dN_t\right)\right] \leq E_c\left[\int_0^{\infty} e^{-\rho t}(C_1^* + K)dN_t\right] = \frac{\lambda}{\lambda+\rho}(C_1^* + K)$$

and it follows that $\pi^U \in \Pi$. On the other hand, using an argument similar to that of the proof of Proposition 1 it can be shown that the process

$$Y_t = e^{-\rho t \wedge \tau_0 \wedge \tau_U^*} U(C_{t\wedge\tau_0\wedge\tau_U^*}) + \int_0^{t\wedge\tau_0\wedge\tau_U^*} e^{-\rho t}\left(dD_s^U - f_{s-}^U dN_s\right)$$

is a uniformly integrable martingale. An application of the optional sampling theorem at the finite stopping time τ_N then implies

$$\begin{aligned} U(c) = Y_0 = E[Y_{\tau_N}] &= E_c\left[e^{-\rho\tau\wedge\tau_0}U(C_{\tau\wedge\tau_0}) + \int_0^{\tau\wedge\tau_0} e^{-\rho t}\left(dD_s^U - f_{s-}^U dN_s\right)\right] \\ &= E_c\left[1_{\{\tau<\tau_0\}}e^{-\rho\tau}V_1(C_\tau) + 1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_0 + \int_0^{\tau\wedge\tau_0} e^{-\rho t}\left(dD_s^U - f_{s-}^U dN_s\right)\right] \end{aligned}$$

and the desired result now follows from Lemma 18 by taking the supremum over the set of admissible strategies on both sides. ■

Lemma 24 *Let q denote an arbitrary function and define $\hat{q}$ implicitly through*

$$q(c) = F_0(c)\hat{q}(Z(c)) = F_0(c)\hat{q}\left(\frac{G_0(c)}{F_0(c)}\right).$$

Then we have:

(a) *The function Z is monotone increasing and $\hat{q}(y) = q(Z^{-1}(y))/F_0(Z^{-1}(y))$,*

(b) *The function q solves (42) if and only if the function $\hat{q}$ is linear,*

(c) *For an arbitrary $c \geq 0$,*

$$\min\{\hat{q}'(y)(q(c)/F_0(c))', \hat{q}''(y)(\mathcal{L}_0 q(c) - \lambda q(c)))\} \geq 0$$

with $y = Z(c)$.

Proof. The first two claims follow by direct calculation using the definition of $\hat{q}$, F_0 and G_0. The third claim is formula (6.2) in Dayanik and Karatzas (2003). ■

Lemma 25 *The threshold $C_W^* = C_W^*(K)$ is decreasing in K and satisfies $C_W^*(K^*) = C_0^*$.*

Proof. By (22), we have that C_W^* is the unique solution X to

$$\ell_0(0) = \tilde{V}_0(0;X) + \frac{\lambda}{\lambda+\rho}\left((r/\rho - 1)\, C_1^* - K + \frac{\mu_1}{\rho} + \frac{\mu_0}{\lambda+\rho-r}\right)$$

where $\tilde{V}$ is defined in (48). As shown in the proof of Lemma 12, the function $\tilde{V}_0(0;X)$ is monotone decreasing in X and the desired monotonicity with respect to K thus follows by differentiation. To show that C_W^* converges to C_0^* as $K \to K^*$ we argue as follows. By definition of K^* we have

$$V_1(C_1^*) - C_1^* - K^* = V_0(C_0^*) - C_0^*.$$

Thus, it follows from Lemma 16 that the function V_0 solves

$$0 = \mathcal{L}_0 V_0(c) + \lambda[V_0(C_0^*) - C_0^* + c - V_0(c)] = \mathcal{L}_0 V_0(c) + \lambda[V_1(C_1^*) - C_1^* - K^* + c - V_0(c)]$$

on the interval $[0, C_0^*]$ with the boundary conditions $V_0'(C_0^*) = 1$, $V''(C_0^*) = 0$ and the desired result follows from the uniqueness part of Lemma 21. ■

Lemma 26 *The following are equivalent:*

(1) $K > K^*$,

(2) $W(C_W^*(K)) - C_W^*(K) < V_1(C_1^*) - (C_1^* + K)$.

Proof. Evaluating the ODE

$$\mathcal{L}_0 W(c) + \lambda[V_1(C_1^*) - C_1^* - K + c] = 0$$

at the point $c = C_W^*$ and using the definition of K^* we obtain that

$$W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K) = \frac{\rho}{\lambda+\rho}(K - K^*) + \frac{\rho - r}{\lambda+\rho}(C_0^* - C_W^*)$$

and the desired equivalence now follows from Lemma 25. ■

Lemma 27 (a) *If $K \geq K^*$ then $W(c) \geq V_1(c-K)$ for all $c \geq K$.*

(b) *If $K < K^*$ then either $V_1(c-K) \geq W(c)$ for all $c \geq K$, or there exists a unique crossing point $C_1^* \leq \tilde{C} \leq C_1^* + K$ such that $V_1(c-K) < W(c)$ if and only if $c < \tilde{C}$.*

Proof. We only prove part (b) as both claims follow from similar arguments. Since W is concave by Lemma 21, we have

$$W(c) \leq W(C_W^*) + c - C_W^*$$

and it now follows from Lemma 26 that

$$k(c) \equiv W(c) - V_1(c-K) \leq W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K) \leq 0.$$

for all $c \geq C_1^* + K$. In order to complete the proof of the first part we distinguish three cases depending on the location of the threshold C_W^*.

CASE 1: $C_W^* \leq K$. In this case the function W is linear for $c \geq K$. Since V_1 is concave, the functions $V_1(c-K)$ and $W(c)$ can have at most 2 crossing points. But, since $V_1(c-K) < W(c)$ for large c as shown above, there can be at most one crossing point.

CASE 2: $C_W^* \geq C_1^* + K$. Suppose towards a contradiction that the function k has more than one zero and denote by $z_0 \leq z_1$ its two largest zeros in the interval $[K, C_1^* + K]$. Then, $k(c) > 0$ for $c \in (z_0, z_1)$ due to the above inequality and it follows that the function k has a positive local maximum in the open interval (z_0, z_1). Since $C_W^* \geq C_1^* + K$, it follows from Lemmas 16 and 21 that the function k solves

$$\mathcal{L}_0 k(c) - \lambda k(c) + (-\mu_1 + \mu_0 + rK) V_1'(c-K) = 0 \tag{54}$$

in the interval $[0, C_1^* + K]$ and the required contradiction now follows from Lemma 13 and the fact that $\mu_1 - \mu_0 - rK > 0$ whenever $K \leq K^*$ as a result of Lemma 20.

CASE 3: $C_W^* \in [K, C_1^* + K]$. If $z_1 \leq C_W^*$ then the argument of Case 2 above still applies so assume that the function k does have zeros in the interval $[C_W^*, C_1^* + K]$. Since $V_1(c-K)$ is concave in that interval and $k(C_1^* + K) \leq 0$ we know that the function k can have at most one zero there. Denote the location of this zero by $\bar{z}$ so that $k(c) > 0$ for $c \in [C_W^*, \bar{z})$ and $k(c) \leq 0$ for $c \geq \bar{z}$. Since the function k solves (54) on the interval $[0, C_W^*]$ and satisfies $k(C_W^*) > 0$, $k'(C_W^*) = 1 - V_1'(C_W^* - K) < 0$ it follows from Lemma 14 that $k(c) > 0$ for all $c \leq C_W^*$ and the proof is complete. ■

B. Proof of Theorem 4

We start this appendix with a standard verification result for the HJB equation associated with the firm's problem:

Lemma 28 *If ϕ is continuous, piecewise twice continuously differentiable and such that*

$$\max\{\mathcal{L}_0\phi(c) + \mathcal{F}\phi(c); 1 - \phi'(c); V_1(c-K) - \phi(c), \ell_0(c) - \phi(c)\} \leq 0\,,$$

and at each point c at which $\phi'(c)$ jumps, we have $\phi'(c_-) \geq \phi'(c_+)$. Then, $\hat{V}(c) \leq \phi(c)$ for all $c \geq 0$.

Proof. Fix an arbitrary strategy $\pi \in \Pi$, denote by C_t the corresponding cash buffer process and consider the process

$$Y_t = e^{-\rho t}\phi(C_t) + \int_0^t e^{-\rho s}(dD_s - f_{s-}dN_s).$$

Using arguments similar to those of the proof of Lemma 8 it can be shown that Y_t is a local supermartingale[5] and since

$$Z_t = Y_{t\wedge\tau_0} \geq -\int_0^{\tau_0} e^{-\rho s}(dD_s + f_{s-}dN_s)$$

[5]The cases when the derivative of the function jumps at finitely many points can be handled by the Ito-Tanaka formula. See Karatzas and Shreve (1991, Chapter 3.6) for more details.

where the right hand side is integrable by definition of the set Π we conclude that Z_t is a supermartingale. In particular,

$$\begin{aligned}
\phi(c) &= \phi(C_0) - \Delta\phi(C_0) = Z_0 - \Delta\phi(C_0) \geq E_c[Z_\tau] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau\wedge\tau_0}\phi(C_{\tau\wedge\tau_0}) + \int_{0+}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau\wedge\tau_0}\phi(C_{\tau\wedge\tau_0}) + \int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&\geq E_c\left[1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_i(0) + 1_{\{\tau_0>\tau\}}e^{-\rho\tau}V_1(C_\tau)\right] \\
&+ E_c\left[\int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&\geq E_c\left[1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_i(0) + 1_{\{\tau_0>\tau\}}e^{-\rho\tau}V_1(C_\tau) + \int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] \qquad (55)
\end{aligned}$$

where the first inequality follows from the optional sampling theorem for supermartingales, the second inequality follows from the assumptions of the statement, and the last one follows from

$$\Delta D_0 + \Delta\phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-}-\Delta D_0}^{C_{0-}} (1 - \phi'(c))dc \leq 0\,.$$

Taking the supremum over $\pi \in \Pi$ on both sides of (55) then gives

$$\phi(c) \geq \sup_{\pi\in\Pi} E_c\left[1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_i(0) + 1_{\{\tau_0>\tau\}}e^{-\rho\tau}V_1(C_\tau) + \int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right]$$

and the result now follows from Lemma 18. ■

Lemma 29 *If $U'(0) \geq W'(0)$ then U satisfies the conditions of Lemma 28 and $C_U^* \leq C_1^* + K$.*

Proof. First of all, if $\psi'(K;K) \geq V_1'(0)$, then we have $C_U^* = K$ from Lemma 23 and we only need to show that $U'(c) = \psi'(c;K) \geq 1$ for $c \leq K$. To this end, let $\bar{W}$ be the unique solution to

$$\mathcal{L}_0\bar{W}(c) - \lambda\bar{W}(c) + \lambda(V_1(C_1^*) - C_1^* + c) = 0, \qquad c \geq 0,$$

which coincides with the function W on the interval $[0, C_W^*]$. Since $\bar{W}$ satisfies $\bar{W}'(C_W^*) = 1$ and $\bar{W}''(C_W^*) = 0$, it follows from Lemma 15 that $\bar{W}'(c) \geq \bar{W}'(C_W^*) = 1$ for all $c \geq 0$. Then, the difference $m(c) = \psi(c;K) - \bar{W}(c)$ satisfies

$$\mathcal{L}_0 m(c) - \lambda m(c) = 0, \qquad c \in [0, K]\,. \qquad (56)$$

Furthermore, $m(0) = 0$ and $m'(0) \geq 0$ since, by assumption $U'(0) = \psi'(0;K) \geq W'(0) = \bar{W}'(0)$. Lemma 14 implies that $m'(c) \geq 0$ that is $\psi'(c;K) \geq \bar{W}'(c) \geq 1$, which is what had to be proved.

Now assume that $\psi'(K;K) < V_1'(0)$ so that $C_U^* > K$. In order to show that $U'(c) \geq 1$, consider the function $\phi(c) = U(c) - \bar{W}(c)$. By Lemmas 21, 23 we have that the function ϕ solves (56) and,

since $\phi(0) = 0$ and $\phi'(0) = U'(0) - W'(0) > 0$ by assumption, it follows from Lemma 13 that $\phi'(c) \geq 0$ and consequently

$$U'(c) \geq \bar{W}'(c) \geq 1, \qquad c \leq C_U^*. \tag{57}$$

Using (57) in conjunction with the definition of the liquidation value, it is immediate to show that

$$U(c) = U(0) + \int_0^c U'(x)dx = \ell_0(0) + \int_0^c U'(x)dx \geq \ell_0(0) + c = \ell_0(c).$$

The inequality $U(c) \geq V_1(c - K)$ is contained in the proof of Lemma 23 and the proof that U satisfies the conditions of Lemma 28 will be complete once we show that $\mathcal{L}_0 U(c) + \mathcal{F}U(c) \leq 0$. A direct calculation using the fact that, as shown below, $C_U^* \leq C_1^* + K$ together with the definition and concavity of the functions U and V_1 shows that

$$\mathcal{L}_0 U(c) + \mathcal{F}U(c) = \begin{cases} 0, & c \leq C_U^*, \\ (rK - \mu_1 + \mu_0)V_1'(c - K), & C_U^* \leq c \leq C_1^* + K, \\ (r - \rho)(c - (C_1^* + K)) + \mu_0 - \mu_1 + rK, & c \geq C_1^* + K. \end{cases}.$$

and the desired result now follows from the increase of V_1 and the fact that $\mu_0 - \mu_1 + rK < 0$ for all $K \leq K^*$ by Lemma 20.

In order to show that $C_U^* \leq C_1^* + K$ assume towards a contradiction that $C_U^* > C_1^* + K$. In this case we have that $U'(C_U^*) = 1$ and, since $U(c) > V_1(c - K)$, we get that $U(c)$ is convex in a small neighborhood of C_U^*. This implies that $U'(c) < U'(C_U^*) = 1$ for c in this small neighborhood, which is impossible die to the first part of the proof. ■

Having dealt with the case where the firm uses exclusively the strategy (U), we now turn to the case in which the firm mixes the strategies (U) and (W). To state the result, recall that the function v_1 is defined by equation (50).

Lemma 30 *Assume that $U'(0) < W'(0)$. Then the unique solution to the free boundary problem defined by (23), (18), (24), (25), (26) is given by*

$$V(c) = \begin{cases} W(c), & c \leq C_L^*, \\ S(c), & C_L^* \leq c \leq C_H^*, \\ V_1(c - K), & c \geq C_L^*, \end{cases} \tag{58}$$

where the function S is defined by (28) and the constants $C_L^* \in [C_W^*, \tilde{C})$, $C_H^* \in [C_U^*, C_1^* + K]$ are the unique solutions to the value matching and smooth pasting condition

$$\begin{aligned} S(C_L^*) &= \xi_G(C_H^*)F_0(C_L^*) - \xi_F(C_H^*)G_0'(C_L^*) + \Phi(C_L^*) \\ &= W(C_L^*), \end{aligned} \tag{59}$$

$$\begin{aligned} S'(C_L^*) &= \xi_G(C_H^*)F_0'(C_L^*) - \xi_F(C_H^*)G_0'(C_L^*) + \Phi'(C_L^*) \\ &= W'(C_L^*) = 1. \end{aligned} \tag{60}$$

Furthermore, $\max\{W(c), V_1(c - K)\} \leq V(c) \leq \hat{V}(c)$ *for all* $c \geq 0$.

Proof. Using arguments similar to those of the proof of Lemma 9, it can be shown that the unique solution to (23) such that (24) and (26) holds is given by (58) and the first part of the proof will be complete once we show that (59) and (60) admit unique solutions.

By Lemma 24, finding a solution to (18), (23), (24), (25), (26) is equivalent to finding a linear function that is tangent to the graph of the functions $\hat{w}$ and $\hat{v}_1$ defined by

$$w(c) = W(c) - \Phi(c) = F_0(c)\hat{w}(Z(c)) = F_0(c)\hat{w}\left(\frac{G_0(c)}{F_0(c)}\right).$$

and (52). A direct calculation using the results of Lemma 21 shows that

$$\mathcal{L}_0 w(c) - \lambda w(c) = (r-\rho)(c - C_W^*)^+$$

and it now follows from Lemma 24 that the function $\hat{w}$ is linear for $y \le Z(C_W^*)$ and concave otherwise. Since $W(c)$ is concave by Lemma 21, we get

$$w'(c) = W'(c) - \frac{\lambda}{\lambda+\rho-r} \ge W'(C_W^*) - \frac{\lambda}{\lambda+\rho-r} = \frac{\rho-r}{\lambda+\rho-r} > 0.$$

Since F_0 is nonnegative and decreasing, the ratio $w(c)/F_0(c)$ is positive and strictly increasing for sufficiently large c. Therefore, Lemma 24 implies that $\hat{w}$ is increasing for sufficiently large values of y and, since $\hat{w}(y)$ is concave, it is globally increasing.

Since

$$U(0) = W(0) = \ell_0,$$

we obtain that both $\hat{w}$ and the function

$$\hat{u}(y) = (U(Z^{-1}(y)) - \Phi(Z^{-1}(y)))/F_0(Z^{-1}(y))$$

are linear on $[Z(0), Z(C_W^*) \wedge Z(C_U^*)]$ and coincide at $Z(0)$. The inequality $U'(0) < W'(0)$ implies that $\hat{u}(y) \le \hat{w}(y)$ for all $y \in [Z(0), Z(C_W^*) \wedge Z(C_U^*)]$. It follows that $C_W^* \le \tilde{C} < C_U^*$ because $\hat{w}$ is a linear function that crosses the graph of the concave function $\hat{v}_1(y)$ at the point $Z(\tilde{C})$ and, by the definition of $\tilde{C}$, we have $\hat{w}(y) < \hat{v}_1(y)$ for all $y > Z(\tilde{C})$.

Since

$$U(c) \ge V_1(c-K), \qquad c \le C_U^*,$$

by Lemma 23, we get that the linear function

$$\bar{w}(y) = \frac{\hat{w}(Z(C_W^*)) - \hat{w}(Z(0))}{Z(C_W^*) - Z(0)} y + \frac{\hat{w}(Z(0))Z(C_W^*) - \hat{w}(Z(C_W^*))Z(0)}{Z(C_W^*) - Z(0)}$$

is tangent to the concave function $\hat{w}(y)$ and lies strictly above the concave function $v_1(y)$ for all $y \ge Z(\tilde{C})$. On the other hand, since

$$\hat{v}_1(Z(\tilde{C})) = \hat{w}(Z(\tilde{C})),$$
$$\hat{v}_1'(Z(\tilde{C})) > \hat{w}'(Z(\tilde{C}))$$

as a result of Lemma 27, we have that the tangent line to $\hat{w}$ at the point $y = Z(\tilde{C})$ lies strictly below v_1 for $y > Z(\tilde{C})$. By continuity, this implies that there exists a unique point $y_L^* \in (Z(C_W^*), Z(\tilde{C}))$ such that the tangent line to $\hat{w}$ at y_L^* is also tangent to $\hat{v}_1$ at some $y_H^* > y_L^*$. Setting

$$C_L^* = Z^{-1}(y_L^*) < Z^{-1}(y_H^*) = C_H^*$$

produces the unique solution to (59), (60) and it now only remains to show that $C_U^* < C_H^* < C_1^* + K$. Since $\hat{w}$ is is increasing and concave its tangent line at the point y_L^* crosses the vertical axis above the level $\hat{w}(Z(0))$. However, if C_U^* was larger than C_H^* then this tangent would have to cross the vertical axis below $\hat{u}(Z(0)) = \hat{w}(Z(0))$ thus leading to a contradiction. Furthermore, since $\hat{w}$ and $\hat{v}_1$ are both concave, we get that $\hat{v} \geq \max\{\hat{w}, \hat{v}_1\}$ and therefore $V(c) \geq \max\{W(c)\,,\, V_1(c-K)\}$. The claim $C_H^* > C_1^* + K$ follows from Lemma 31 below.

In order to show that $V(c) \leq \hat{V}(c)$, and thus complete the proof, let τ_L^* (resp. τ_H^*) denote the first time that the firm's cash reserves falls below C_L^* (resp. increases above C_H^*). Using arguments similar to those of the proof of Lemma 29 it can be shown that

$$V(c) = E_c\Big[1_{\{\tau_H^* < \tau_N \wedge \tau_L^*\}} e^{-\rho\tau_H^*} V_1(C_H^* - K) + 1_{\{\tau_L^* < \tau_N \wedge \tau_H^*\}} e^{-\rho\tau_L^*} W(C_L^*) \\ + 1_{\{\tau_N < \tau_L^* \wedge \tau_H^*\}} e^{-\rho\tau_N}(V_1(C_1^*) - C_1^* - K + C_{\tau_N -})\Big].$$

On the other hand, using arguments similar to those of the proof of Proposition 1 it can be shown that the function W satisfies

$$W(c) = E_c\left[1_{\{\tau_0 < \tau_N\}} e^{-\rho\tau_0}\ell_0 + 1_{\{\tau_0 \geq \tau_N\}} e^{-\rho\tau_N} V_1(C_1^*) + \int_0^{\tau_0 \wedge \tau_N} e^{-\rho s}(dL_s - f_{s-}^U dN_s)\right]$$

where $L_t = \sup_{s \leq t}(X_t - C_W^*)^+$ with

$$dX_t = (rX_{t-} + \mu_i)dt + \sigma dB_t + (C_W^* - X_{t-})^+ dN_t\,,$$

and f^U is defined as in equation (53). Combining these two equalities and using the law of iterated expectations then gives

$$V(c) = E_c\left[1_{\{\tau_0 < \tau\}} e^{-\rho\tau_0}\ell_0 + 1_{\{\tau_0 \geq \tau\}} e^{-\rho\tau} V_1(C_\tau) + \int_0^{\tau_0 \wedge \tau} e^{-\rho s}(dD_s^V - f_{s-}^U dN_s)\right]$$

where $\tau = \tau_N \wedge \tau_H^*$ and the cumulative dividend process is defined by

$$D_t^V = \int_0^t 1_{\{C_{s-} \leq C_L^*\}} dL_s.$$

As is easily seen the strategy (τ, D^V, f^U) is admissible and the desired result now follows from Lemma 18 by taking the supremum over admissible strategies on both sides. ■

Lemma 31 *If $U'(0) < W'(0)$, then V satisfies the conditions of Lemma 28 and $C_H^* < C_1^* + K$.*

Proof. To show that $V' \geq 1$ we start by observing that this inequality holds in $[0, C_L^*] \cup [C_H^*, \infty)$ due to the definition of the function V and Lemmas 16, 21, 23. On the other hand, since we know that $C_H^* \geq C_L^* \geq C_W^*$, we have $V'(C_L^*) = W(C_L^*) = 1$ and

$$V(c) \geq W(c) = W(C_W^*) + c - C_W^*, \qquad C_L^* \leq c \leq C_H^*.$$

This immediately implies that $V''(C_L^*) \geq 0$ and since $J(c) = V'(c)$ is a solution to

$$\frac{\sigma^2}{2} J''(c) \; + \; (rc + \mu_0) J'(c) - (\lambda + \rho - r) J(c) + \lambda \; = \; 0,$$

it follows from Lemma 13 that $J' = V''$ can have at most one zero in the interval $\mathcal{I} = [C_L^*, C_H^*]$. If no such zero exists then $V'' \geq 0$ in $\mathcal{I}$ and consequently $V'(c) \geq V'(C_L^*) = 1$ for all $c \in \mathcal{I}$. If on the contrary V'' has one zero located at some $c^* \in \mathcal{I}$ then we have that V' reaches a global maximum over $\mathcal{I}$ at the point c^* and since $V'(C_H^*) = V_1'(C_H^* - K) \geq 1$ due to the concavity of V_1, we conclude that the inequality $V'(c) \geq 1$ holds for all $c \in \mathcal{I}$.

Let us now show that $C_H^* \leq C_1^* + K$. Indeed, if this is not the case, we have $V'(C_H^*) = 1$. Since $V(c) > V_1(c - K)$, we have that $V(c)$ is convex in a small neighborhood of C_H^* and therefore $V'(c) < V'(C_H^*) = 1$ for c in this small neighborhood, which is impossible by the previous paragraph.

Using the fact that $V' \geq 1$ in conjunction with the definition of the liquidation value it is immediate to show that

$$V(c) = V(0) + \int_0^c V'(x)dx = \ell_0(0) + \int_0^c V'(x)dx \geq \ell_0(0) + c = \ell_0(c).$$

The fact that $V(c) \geq \max\{W(c)\,,\, V_1(c-K)\}$ is contained in Lemma 30. Finally, since $C_W^* \leq C_L^* \leq C_H^* \leq C_1^* + K$ it follows from the definition and concavity of the functions W and V_1 that

$$\mathcal{L}_0 V(c) + \mathcal{F} V(c) = \begin{cases} 0, & c \leq C_W^*, \\ (r-\rho)(C - C_W^*), & C_W^* \leq c \leq C_L^*, \\ 0, & C_L^* \leq c \leq C_H^*, \\ A V_1'(c-K) + (r-\rho)(c - C_1^* - K)^+, & c \geq C_H^*, \end{cases}$$

where we have set $A = \mu_0 - \mu_1 + rK$. By Lemma 19 we know that $A \geq 0$ whenever $K \leq K^*$ and it thus follows from the increase of the function V_1 that

$$\mathcal{L}_0 V(c) + \mathcal{F} V(c) \leq 0, \qquad c \geq 0.$$

and the proof is complete. ■

Lemma 32 *There exists a unique $K^{**} \in (0, K^*)$ such that $U'(0) > W'(0)$ if and only if $K < K^{**}$.*

Proof. We know from the proof of Lemma 33 that $U'(0) = \psi'(0; K) > W'(0)$ for sufficiently small K. Similarly, for $K = K^*$ we know that $V_1(c - K^*)$ touches $W(c)$ from below at $C_1^* + K$ so that $C_L^* = C_H^* = C_1^* + K$ and $U'(0) < W'(0)$. Thus, it suffices to show that there exists a unique point $K^{**} \leq K^*$ such that

$$U'(0; K^{**}) = W'(0; K^{**})\,.$$

Assume towards a contradiction that this is not the case so that there exist two points $K_1 < K_2$ such that $U'(0;K_1) = W'(0;K_1)$ and $U'(0;K_2) = W'(0;K_2)$. Let the function $\bar{W}_i$ denote the unique solution to

$$\mathcal{L}_0\bar{W}_i(c) - \lambda\bar{W}_i(c) + \lambda(V_1(C_1^*) - C_1^* - K_i + c) = 0, \qquad c \geq 0,$$

which coincides with the function $W(\cdot;K_i)$ on the interval $[0, C_W^*(K_i)]$ and recall from the proof of Lemma 29 that this function is concave for $c \leq c_i^* = C_W^*(K_i)$ and convex for $c \geq c_i^*$ so that it satisfies $\bar{W}_i'(c) > \bar{W}'(c_i^*) = 1$ for all $c \neq c_i^*$. Since $U(0;K_i) = \bar{W}_i(0)$ by definition, the equality $U'(0;K_i) = W'(0;K_i)$ implies that the functions coincide for $c \leq C_U^*(K_i)$. Consider the function

$$m(c) = \bar{W}_i(c) - V_1(c - K_i)$$

and which satisfies equation (54). If $C_U^*(K_i) > K_i$, then $m(C_U^*(K_i)) = m'(C_U^*(K_i)) = 0$ and it follows from the proof of Lemma 29 that we have $m(c) \geq 0$ for all $c \geq K_i$. If $C_U^*(K_i) = K_i$ then $\bar{W}_i'(C_U^*(K_i)) \geq V_1'(0)$ implies that we have $m(C_U^*(K_i)) = 0$ as well as $m'(C_U^*(K_i)) \geq 0$ and therefore $m(c) \geq 0$ for $c \geq K_i$ by Lemma 14.

Now consider the function

$$k(c) = \bar{W}_2'(c) - \bar{W}_1'(c)$$

which is a solution to

$$(rc + \mu_0)k'(c) + \frac{\sigma^2}{2}k''(c) - (\rho + \lambda - r)k(c) = 0, \qquad c \geq 0,$$

and satisfies $k(c_2^*) > 0$, $k(c_1^*) < 0$. Since $k(c)$ cannot have local negative minima by Lemma 13, it follows that there exists a unique point $c_* \in (c_2^*, c_1^*)$ such that $k(c_*) = 0$, $k'(c_*) > 0$ and $k(c) > 0$ for all $c > c_*$ and $k(c) < 0$ for $c < c_*$. That is, $\bar{W}_2 - \bar{W}_1$ attains its global minimum at c_* and $(\bar{W}_2 - \bar{W}_1)''(c_*) > 0$. Evaluating the equation

$$\frac{1}{2}\sigma^2\,(\bar{W}_2 - \bar{W}_1)''(c) + (rc + \mu_0)(\bar{W}_2 - \bar{W}_1)'(c) - (\rho + \lambda)(\bar{W}_2 - \bar{W}_1)(c) + \lambda(K_1 - K_2) \;=\; 0$$

at the point $c = c_*$, we get

$$(\bar{W}_2 - \bar{W}_1)(c_*) \;>\; \frac{\lambda}{\rho + \lambda}(K_1 - K_2)\,,$$

and therefore

$$(\bar{W}_1 - \bar{W}_2)(c) \;<\; \frac{\lambda}{\rho + \lambda}(K_2 - K_1)$$

for all c. However, since by the above, $\bar{W}_1(c) \;\geq\; V_1(c - K_1)$ for $c \geq K_1$ and $V_1' \geq 1$, we get

$$\begin{aligned}\frac{\lambda}{\rho + \lambda}(K_2 - K_1) \;&\geq\; W_1(C_H^*) - W_2(C_H^*) = W_1(C_H^*) - V_1(C_H^* - K_2) \\ &\geq V_1(C_H^* - K_1) - V_1(C_H^* - K_2) \geq K_2 - K_1,\end{aligned}$$

which is a contradiction. ■

Lemma 33 *There exists a unique* $\underline{K} \in (0, K^{**})$ *such that, for* $K \in (0, K^{**})$, *we have* $\psi'(K, K) < V_1'(0)$ *if and only if* $K > \underline{K}$.

Proof. First of all, we show that $\lim_{K \downarrow 0} \psi'(K; K) = +\infty$. Indeed,

$$G_0(K)F_0(0) - F_0(K)G_0(0) \;\approx\; K\,(G_0'(0)F_0(0) - F_0'(0)G_0(0)) =: \alpha K$$

with $\alpha > 0$. Therefore,

$$\psi'(K; K) \;\approx\; (\alpha K)^{-1}(G_0(0)(\ell_0 - \ell_1)F_0'(0) - F_0(0)(\ell_0 - \ell_1)G_0'(0)) = K^{-1}(\ell_1 - \ell_0)$$

and the required assertion follows from the fact that $\ell_1 \geq \ell_0$.

Since $\psi'(K; K)$ is continuous in K, it remains to show that the equation $\psi'(K; K) = V_1'(0)$ can have at most one solution $\underline{K}$. Suppose the contrary that $K_1 < K_2$ are tow solutions of that equation and let $\psi_i(c) = \psi(c; K_i)$ so that

$$\mathcal{L}_0\psi_i(c) - \lambda\psi_i(c) + \lambda(V_1(C_1^*) - C_1^* - K_i + c) \;=\; 0\;,\; c \geq 0.$$

By Lemma 29, $\psi'(c; K) \geq 1$ since $K \leq K^{**}$. Now, consider the functions $\tilde{\psi}_i(y) = \psi_i(y + K_i)$. Then,

$$\frac{\sigma^2}{2}\tilde{\psi}_i''(y) + (ry + rK_i + \mu_0)\tilde{\psi}_i'(y) - (\rho + \lambda)\tilde{\psi}_i(y) + \lambda(V_1(C_1^*) - C_1^* + y) \;=\; 0\,.$$

Let $m(c) = \tilde{\psi}_1(y) - \tilde{\psi}_2(y)$. Then, $m(0) = \psi_1(K_1) - \psi_2(K_2) = 0$, $m'(0) = \psi_1'(K_1) - \psi_2'(K_2) = 0$. Furthermore,

$$\frac{\sigma^2}{2}m''(y) + (ry + rK_1 + \mu_0)m'(y) - (\rho + \lambda)m(y) + r(K_1 - K_2)\tilde{\psi}_2'(y) = 0\,.$$

Since $\tilde{\psi}_2'(y) > 0$, Lemma 14 implies that the function m is positive and monotone decreasing so that $\tilde{\psi}_1(y) > \tilde{\psi}_2(y)$ for $y < 0$. Since $\tilde{\psi}_2$ is monotone decreasing, this further implies that

$$\ell_0 = \tilde{\psi}_1(-K_1) > \tilde{\psi}_2(-K_1) > \tilde{\psi}_2(-K_2) = \ell_0,$$

which is a contradiction. ■

C. Issuance Costs

To facilitate the proof, we redefine the operator $\mathcal{F}$ of equation (38) as

$$\mathcal{F}V(c) \;=\; \lambda \max_{f \geq 0}\,(V(c + f) \;-\; (1 + \varepsilon)\,f \;-\; V(c) \;-\; I_{f>0}\,\kappa)\,.$$

As in the text we denote by μ the growth rate of the firm's cash flows and accordingly drop the subscript i from the definition of all functions and operators.

Proof of Propositions 6 and 7. In complete analogy with Proposition 1, it suffices to show that $V(c)$ solves the HJB equation

$$\max\{\mathcal{L}\phi(c) + \mathcal{F}\phi(c), 1 - \phi'(c), \ell(c) - \phi(c)\} = 0\,. \tag{61}$$

Suppose first that $\varepsilon \geq \bar{\varepsilon}$. Then, the concave function $V_{\lambda=0}(c)$ clearly solves (61) by Proposition 1 and it is optimal not to raise any funds. Similarly, if $\varepsilon < \bar{\varepsilon}$ but $\kappa \geq \overline{\kappa}(\varepsilon)$, the same conclusion applies and $V_{\lambda=0}(c)$ is the value function. So, let us assume that $\varepsilon < \bar{\varepsilon}$ and $\kappa < \overline{\kappa}(\varepsilon)$. Then, by the definition of $\bar{\varepsilon}$ and $\overline{\kappa}(\varepsilon)$, the thresholds $\underline{C}(C^*_{\lambda=0})$ and $\overline{C}(C^*_{\lambda=0})$ exist and are positive. We will first show that the following is true.

Lemma 34 *$\underline{C}(X)$ and $\overline{C}(X)$ are monotone increasing in X.*

Proof. Let us first show that, for $X_1 < X_2$, we have

$$V'_{\lambda=0}(c; X_1) \;\leq\; V'_{\lambda=0}(c; X_2) \tag{62}$$

for all c. Indeed, by equation (2), we have that $k(c) = V_{\lambda=0}(c; X_2) - V_{\lambda=0}(c; X_1)$ solves equation (30) for $c \leq X_1$ with $k'(X_1) = V'_{\lambda=0}(c; X_2) - 1 \;>\; 0$ and $k''(X_1) = V''_{\lambda=0}(c; X_2) < 0$. By Lemma 15, this implies that $k'(c) \geq 0$, $k''(c) \leq 0$ for all $c \leq X_1$. Finally, for $c \geq X_1$, $k'(c) = V'_{\lambda=0}(c; X_2) - 1 \geq 0$, and the claim follows. Thus, we have

$$V'_{\lambda=0}(\overline{C}(X_1); X_2) \;\geq\; V'_{\lambda=0}(\overline{C}(X_1); X_1) = 1 + \varepsilon,$$

and therefore $\overline{C}(X_2) > \overline{C}(X_1)$ since $V_{\lambda=0}$ is concave. On the other hand, since $V'_{\lambda=0}(c; X_2) \geq V'_{\lambda=0}(c; X_1)$ for all $c \geq 0$, we have

$$\begin{aligned}
\kappa \;&=\; V_{\lambda=0}(\overline{C}(X_1); X_1) - V_{\lambda=0}(\underline{C}(X_1); X_1) - (1+\varepsilon)\,(\overline{C}(X_1) - \underline{C}(X_1)) \\
&\leq\; V_{\lambda=0}(\overline{C}(X_1); X_2) - V_{\lambda=0}(\underline{C}(X_1); X_2) - (1+\varepsilon)\,(\overline{C}(X_1) - \underline{C}(X_1)) \\
&\leq\; V_{\lambda=0}(\overline{C}(X_2); X_2) - V_{\lambda=0}(\underline{C}(X_2); X_2) - (1+\varepsilon)\,(\overline{C}(X_2) - \underline{C}(X_1))
\end{aligned}$$

and therefore $\underline{C}(X_2) \geq \underline{C}(X_1)$. ■

By Lemma 12, we know that the function $V_{\lambda=0}(c; X)$ is monotone decreasing with respect to the threshold X for all $c < X$. Therefore,

$$V_{\lambda=0}(\overline{C}(X); X) - (1+\varepsilon)\overline{C}(X) \;=\; V_{\lambda=0}(\underline{C}(X); X) - (1+\varepsilon)\underline{C}(X) + \kappa \tag{63}$$

is monotone decreasing in X. Indeed,

$$\frac{d}{dX}(V_{\lambda=0}(\overline{C}(X); X) - (1+\varepsilon)\overline{C}(X)) \;=\; \frac{\partial}{\partial X} V_{\lambda=0}(\overline{C}(X); X) \;<\; 0\,. \tag{64}$$

Now, since $V(c; X)$ is a solution to equation (29) for $c \leq \underline{C}(X)$ it follows from by the same argument as in the proof of Proposition 1 that it can be written down as

$$V(c; X) \;=\; \tilde{V}(c; X) + \frac{\lambda\,(1+\varepsilon)\,c}{\rho + \lambda - r} + \frac{\lambda\left((\rho + \lambda - r)\,(V_{\lambda=0}(\underline{C}(X); X) - (1+\varepsilon)\underline{C}(X)) \;+\; \mu\right)}{(\rho + \lambda - r)\,(\rho + \lambda)}\,, \tag{65}$$

where $\tilde{V}$ solves the homogeneous ODE given by (30). A direct calculation shows that, in fact, V satisfies a high contact condition with $V_{\lambda=0}$ at $\underline{C}(X)$. Therefore, by Lemma 15, V is concave and increasing on $[0, \underline{C}(X)]$ and, in particular, $V'(c; X) \geq 1 + \varepsilon$ for all $c \leq \underline{C}(X)$.

By (63) and (64), $V_{\lambda=0}(\underline{C}(X); X) - (1+\varepsilon)\underline{C}(X)$ is monotone decreasing in X. We will now prove the following auxiliary result.

Lemma 35 *$\tilde{V}(c;X)$ is concave in c and is monotone decreasing in X.*

Proof. The fact that $\tilde{V}$ is concave in c follows from (65) and the concavity of V so it only remains to prove monotonicity in X. Fix $X_1 < X_2$, define

$$k(c) \;=\; \tilde{V}(c;X_1) - \tilde{V}(c;X_2)$$

and let us first show that $k(\underline{C}(X_1)) \geq 0$. Since $V'_{\lambda=0}(c;X) \geq 1+\varepsilon$ for $c \leq \underline{C}(X)$ and, by Lemma 34, $\underline{C}(X_1) < \underline{C}(X_2)$, we get

$$V_{\lambda=0}(\underline{C}(X_2);X_2) - V_{\lambda=0}(\underline{C}(X_1);X_2) \;\geq\; (1+\varepsilon)\,(\underline{C}(X_2) - \underline{C}(X_1))$$

and combining this with equation (65) gives

$$\begin{aligned}
\tilde{V}(\underline{C}(X_1);X_1) \;-\; \tilde{V}(\underline{C}(X_1);X_2) \;=\; &\Bigg(V_{\lambda=0}(\underline{C}(X_1);X_1) \\
&- \frac{\lambda}{\rho+\lambda}(V_{\lambda=0}(\underline{C}(X_1);X_1) - (1+\varepsilon)\underline{C}(X_1))\Bigg) - \Bigg(V_{\lambda=0}(\underline{C}(X_1);X_2) \\
&+ \frac{\lambda}{\rho+\lambda}(V_{\lambda=0}(\underline{C}(X_2);X_2) - (1+\varepsilon)\underline{C}(X_2))\Bigg) \;>\; V_{\lambda=0}(\underline{C}(X_1);X_1) \\
&- \frac{\lambda}{\rho+\lambda}(V_{\lambda=0}(\underline{C}(X_1);X_1) - (1+\varepsilon)\underline{C}(X_1)) \;-\; V_{\lambda=0}(\underline{C}(X_2);X_2) \\
&+ (1+\varepsilon)\,(\underline{C}(X_2) - \underline{C}(X_1)) + \frac{\lambda}{\rho+\lambda}(V_{\lambda=0}(\underline{C}(X_2);X_2) - (1+\varepsilon)\underline{C}(X_2)) \\
&= \frac{\rho}{\rho+\lambda}\,((V_{\lambda=0}(\underline{C}(X_1);X_1) - (1+\varepsilon)\underline{C}(X_1)) \;-\; (V_{\lambda=0}(\underline{C}(X_2);X_2) - (1+\varepsilon)\underline{C}(X_2))) \geq 0
\end{aligned} \tag{66}$$

where the last inequality follows from (64).

Suppose towards a contradiction that $\tilde{V}(0;X)$ is not monotone decreasing in X so that there exist $X_1 \leq X_2$ with $\tilde{V}(0;X_1) \;\leq\; \tilde{V}(0;X_2)$. By continuity and (66), there exists a $q < \underline{C}(X_1)$ such that $\tilde{V}(q;X_1) \;=\; \tilde{V}(q;X_2)$. Since k solves equation (30) and clearly satisfies $k(q) = 0$ $k'(q) \geq 0$ and $k(\underline{C}(X_1)) > 0$ a small modification of Lemma 14 implies that $k'(c) \geq 0$ for all $c \in [q, \underline{C}(X_1)]$. Now consider the function defined by

$$p(c) = V(c;X_1) - V(c;X_2)$$

Due to equation 64 we have that this function satisfies $p(q) > 0$ as well as $p'(c) = k'(c) \geq 0$ for all $c \in [q, \underline{C}(X_1)]$. On the other, equation (62) implies that $p'(\underline{C}(X_2)) < 0$ and so there exists a point $z_* \in (\underline{C}(X_1), \underline{C}(X_2))$ where p' will change sign and the minimal value of c at which this happens will be a local maximum of p but since

$$\frac{1}{2}\sigma^2 p''(c) \;+\; (rc+\mu)\,p'(c) \;-\; \rho\,p(c) \;-\; \lambda\,(V(\underline{C}(X_2)) - V(c) - (1+\varepsilon)(\underline{C}(X_2) - c)) \;=\; 0$$

for all $c \in (\underline{C}(X_1), \underline{C}(X_2))$ it follows from Lemma 13 that p cannot have positive local maxima. This provides the required contradiction and completes the proof. ■

We now complete the proof of Proposition 7. Recall that $V_{\lambda=0}(0; C^*_{\lambda=0}) = \ell(0)$ and consider the function defined by

$$k(c) = V(c; C^*_{\lambda=0}) - V_{\lambda=0}(c; C^*_{\lambda=0}).$$

As is easily seen, we have

$$k(\underline{C}(C^*_{\lambda=0})) = k'(\underline{C}(C^*_{\lambda=0})) = k''(\underline{C}(C^*_{\lambda=0})) = 0 \ < \ k'''(\underline{C}(C^*_{\lambda=0}))\,,$$

so that k is negative and increasing in a small interval to the left of $\underline{C}(C^*_{\lambda=0})$. By the same argument as in the proof of Lemma 14, this implies that $k(c) \leq 0$ for all $c \leq \underline{C}(C^*_{\lambda=0})$ and therefore that $V(0; C^*_{\lambda=0}) < \ell(0)$. Since $\underline{C}(X)$ is monotone increasing in X and always satisfies $\underline{C}(X) < X$, there exists an $X^* \in [0, C^*_{\lambda=0}]$ such that $\underline{C}(X^*) = 0$ and it now follows from (64) that

$$\begin{aligned} V(0; X^*) &= V(\underline{C}(X^*); X^*) - (1+\varepsilon)\underline{C}(X^*) \geq V(\underline{C}(C^*_{\lambda=0}); C^*_{\lambda=0}) - (1+\varepsilon)\underline{C}(C^*_{\lambda=0}) \\ &\geq V(0; C^*_{\lambda=0}) = \ell(0). \end{aligned}$$

By continuity and monotonicity, this implies that the equation $V(0; X) = \ell(0)$ has a unique solution in the interval $[X^*, C^*_{\lambda=0}]$ and completes the proof. ■

D. Time-Varying Capital Supply

In order to describe the solution to the firm's problem in the presence of time-varying capital supply, we need to introduce the following notation. Let

$$\begin{aligned} F_\pm(x) &= M(-0.5\nu_\pm; 0.5; -(rx+\mu)^2/(\sigma^2 r)), \\ G_\pm(x) &= \frac{rx+\mu}{\sigma\sqrt{r}}\ M(-0.5(\nu_\pm - 1); 1.5; -(rx+\mu)^2/(\sigma^2 r)), \end{aligned}$$

where

$$\begin{aligned} \nu_\pm &= \frac{\rho + \hat{\lambda}_\pm}{r}, \\ \hat{\lambda}_\pm &= \frac{(\pi_{LH}+\lambda_L) + (\pi_{HL}+\lambda_H) \pm \sqrt{((\pi_{LH}+\lambda_L) - (\pi_{HL}+\lambda_H))^2 + 4\pi_{LH}\pi_{HL}}}{2}, \end{aligned}$$

and

$$\begin{aligned} \hat{F}_m(x) &= M(-0.5\hat{\nu}_m; 0.5; -(rx+\mu)^2/(\sigma^2 r)), \\ \hat{G}_m(x) &= \frac{rx+\mu}{\sigma\sqrt{r}}\ M(-0.5(\hat{\nu}_m - 1); 1.5; -(rx+\mu)^2/(\sigma^2 r)), \end{aligned}$$

for $m = H, L$, where

$$\hat{\nu}_m \ = \ \frac{\rho + \lambda_m + \pi_{HL}}{r},$$

and as before M is the confluent hypergeometric function. Solving the firm's problem yields the following counterpart to Proposition 1:

Proposition 36 *There exists a unique pair (C_H^*, C_L^*) of cash buffer levels that maximizes the firm value. Let $m \in \{H, L\}$ denote the state with a higher optimal cash level and n the state with the lower optimal cash level.*[6] *For $c < C_n^*$ the value of the firm is given by*

$$\begin{pmatrix} V_L(c) \\ V_H(c) \end{pmatrix} = \sum_{j\in\{\pm\}} \eta_j(\gamma_{1j}F_j(c) + \gamma_{2j}G_j(c)) + \begin{pmatrix} a_L c + b_L \\ a_H c + b_H \end{pmatrix}, \tag{67}$$

with

$$\begin{pmatrix} a_L \\ a_H \end{pmatrix} = \frac{1}{d}\begin{pmatrix} (\rho - r)\lambda_L + \lambda_H\lambda_L + \pi_{HL}\lambda_L + \pi_{LH}\lambda_H \\ (\rho - r)\lambda_H + \lambda_H\lambda_L + \pi_{HL}\lambda_L + \pi_{LH}\lambda_H \end{pmatrix}$$

and

$$\begin{pmatrix} b_L \\ b_H \end{pmatrix} = \frac{1}{\hat{d}}\begin{pmatrix} (\rho + \lambda_H)(\mu a_L + \lambda_L(V_L(C_L^*) - C_L^*)) + Z \\ (\rho + \lambda_L)(\mu a_H + \lambda_H(V_H(C_H^*) - C_H^*)) + Z \end{pmatrix},$$

where

$$\begin{aligned} \hat{d} &= (\rho + \lambda_L + \pi_{LH})(\rho + \lambda_H + \pi_{HL}) - \pi_{LH}\pi_{HL}, \\ d &= (\rho + \lambda_L + \pi_{LH} - r)(\rho + \lambda_H + \pi_{HL} - r) - \pi_{LH}\pi_{HL}, \\ Z &= \pi_{HL}(\mu a_L + \lambda_L(V_L(C_L^*) - C_L^*)) + \pi_{LH}(\mu a_H + \lambda_H(V_H(C_H^*) - C_H^*)), \end{aligned}$$

and we have set

$$\eta_\pm = \begin{pmatrix} \pi_{LH} \\ \pi_{LH} + \lambda_L + \hat{\lambda}_\pm \end{pmatrix}.$$

For $c \in [C_n^*, C_m^*]$, the value of the firm is given by

$$V_m(c) = V_m(C_m^*) + (c - C_m^*),$$

in state m, and

$$\begin{aligned} V_n(c) = {} & a_i\hat{F}_n(c) + b_i\hat{G}_n(c) \\ & + \frac{\lambda_n + \pi_{nm}}{\rho + \lambda_n + \pi_{nm}}\left[\frac{\lambda_n(V_n(C_n^*) - C_n^*) + \pi_{nm}(V_m(C_m^*) - C_m^*)}{\lambda_n + \pi_{nm}} + c + \frac{\mu + rc}{\rho + \lambda_n + \pi_{nm} - r}\right]. \end{aligned}$$

in state n where the constants $\gamma_{1\pm}$, $\gamma_{2\pm}$ a_n and b_n are determined through the corresponding smooth pasting and value matching conditions.

The expressions for the value function, provided in Proposition 36 follow directly from the following lemma which can be verified by direct calculation.

Lemma 37 *The general solution to the non-homogeneous differential system defined by (36) is given by (67) for some constants $\gamma_{1\pm}, \gamma_{2\pm}$.*

[6] While we haven't been able to formally prove that it is the case, economic intuition and the numerical results presented in Figure 6 strongly suggest that $C_H^* > C_i^*$ in which case $m = H$ and $n = L$.

The difficult part in the proof of Proposition 36 consists in finding the unknown constants $\gamma_{1\pm}, \gamma_{2\pm}$ and showing the existence of the optimal cash holdings C_L^*, C_H^*. To this end we will use an iterative procedure, similar to that used in Jiang and Pistorius (2010).

Let Y be an arbitrary but fixed function, denote by τ_Y a random time distributed according to an exponential distribution with parameter $\pi > 0$ and consider the optimal dividend and financing problem defined by

$$V(c) = \sup_{(f,D)\in\Theta} E_c\left[\int_0^{\tau_Y\wedge\tau_0} e^{-\rho t}(dD_t - f_{t-}dN_t) \;+\; 1_{\tau_0<\tau_Y}e^{-\rho\tau_0}\ell_0 \;+\; 1_{\tau_Y<\tau_0}e^{-\rho\tau_Y}Y(C_{\tau_Y})\right] \quad (68)$$

subject to

$$dC_t = (rC_{t-} + \mu_0)dt + \sigma dB_t - dD_t + f_{t-}dN_t$$

where τ_0 stands for the firm's stochastic liquidation time and the set Θ is defined as in Appendix A. This problem is similar to that we considered in Section 3 except that the firm will be automatically liquidated with value $Y(C_{\tau_Y})$ at time τ_Y if it is still alive at that time.

The following sequence of auxiliary lemmas describes how the value function $V(c;Y)$ can be calculated given a conjectured optimal cash reserve level X.

Lemma 38 *Let $X > 0$ denote an arbitrary threshold level. The unique solution to*

$$\frac{1}{2}\sigma^2V''(c) + (rc+\mu)V'(c) - (\rho+\pi)V(c) + \lambda(V(X) - (X-c) - V(c)) + \pi Y(c) \;=\; 0 \quad (69)$$

such that $V'(0;X) = 1$ and $V''(0;X) = 0$ is given by

$$\begin{aligned} V(c;X) = {} & \frac{F(X)G(c) - G(X)F(c)}{J(X)}, \\ & + \frac{rX+\mu+\pi Y(X)+\lambda V(X;X)}{\rho+\lambda+\pi}\frac{G'(X)F(c) - F'(X)G(c)}{W(X)} \\ & + \frac{2}{\sigma^2}\int_c^X \frac{(G(c)F(x) - F(c)G(x))}{W(x)}(\pi Y(x) + \lambda(V(X;X) - (X-c)))dx. \end{aligned}$$

and satisfies

$$V(X;X) \;=\; \frac{rX+\mu+\pi Y(X)}{\rho+\pi}$$

with $J(x) \;=\; G'(x)F(x) - F'(x)G(x)$ and $\nu = (\rho+\lambda+\pi)/r$.

Proof. The proof follows by direct calculation and therefore is omitted. ■

Lemma 39 *Suppose that Y is concave, increasing and satisfies $Y(c) \le k + c$ for some $k > 0$ as well as $Y(0) \ge \ell(0)$. Let also*

$$\eta(c) = \frac{rc+\mu+\pi Y(c)+\lambda(V(X;X)-X) - (\rho+\pi)c}{\rho+\pi+\lambda},$$

and define

$$X^* = \arg\max_{c\geq 0} \eta(c).$$

as well as

$$\hat{V}(c;X) = \begin{cases} V(c;X)\ , & c \geq \zeta(X), \\ V(\zeta(X);X) + (c-\zeta(X))\ , & c \leq \zeta(X). \end{cases}$$

where[7]

$$\zeta(X) \ = \ \max\{\sup\{c < X :\ V'(c;X) < 1\}\,,\, 0\}\,.$$

Then there exists a unique solution $C^* \geq X^*$ such that $\hat{V}(0;C^*) = \ell(0)$. Furthermore, $\zeta(C^*) = 0$ and therefore $V'(c;C^*) \geq 1$ for all $c \in [0, C^*]$.

Proof. It follows by standard arguments that $\hat{V}(c;X)$ is continuous in X. Since X^* is the maximum of the function η we have $\eta'(X^*) = 0 > \eta''(X^*)$.[8] Differentiating equation (69), we get that $V'''(X^*;X^*) < 0$, and therefore $\zeta(X^*) = X^*$. Furthermore, the functions η and

$$\alpha(c) \ = \ \frac{rc + \mu + \pi Y(c) - (\rho+\pi)c}{\rho+\pi}$$

both attain a maximum at the point $c = X^*$ and we have $V(X^*;X^*) = \alpha(X^*)$. Therefore,

$$\hat{V}(0;X^*) = V(X^*;X^*) - X^* = \ \alpha(X^*) - X^* \ \geq \ \alpha(0) \ = \ \frac{\mu + \pi Y(0)}{\rho+\pi} \ \geq \ell(0),$$

since, by assumption, $Y(0) > \ell(0)$. Therefore, it remains to show that $\hat{V}(0;X) \leq \ell(0)$ for sufficiently large values of X. To this end, consider the function defined by

$$\tilde{V}(c;X) = \hat{V}(c;X) - c\,.$$

By direct calculation, it satisfies

$$\frac{1}{2}\sigma^2\tilde{V}''(c;X) \ + \ (rc+\mu)\tilde{V}' \ + \ (\rho+\lambda+\pi)(\eta(c) - \tilde{V}(c)) \ = \ 0 \tag{70}$$

for all $\zeta(X) \leq c \leq X$. Since $X > X^*$ and $\eta(c)$ is concave and attains its global maximum at $c = X^*$ as a result of the assumptions of the statement we have $\eta'(X) < 0$. Differentiating equation (70) at the point $c = X$ and using this property we get

$$\frac{\sigma^2}{2}\tilde{V}'''(X;X) = -(\rho+\pi)\eta'(X) > 0.$$

Since $\tilde{V}'(X) = \tilde{V}''(X) = 0$ we have that the function $\tilde{V}$ is increasing, concave and satisfies $\tilde{V}(c) < \eta(c)$ in a small interval to the left of X. Let θ be the first cash level below X at which the graph of $\tilde{V}$ crosses the graph of η, that is

$$\theta = \max\{c < X : \tilde{V}(c) \geq \eta(c)\}^+,$$

[7] We set $\zeta(X) = 0$ if $V'(c;X) > 1$ for all $c < X$.

[8] Without loss of generality, we assume that the inequality $\eta''(X^*) < 0$ is strict.

and suppose towards a contradiction that the function $\tilde{V}(c)$ is not concave over the interval $[\theta, X]$. Let $c_* = \max\{c < X : \tilde{V}''(c) = 0\}$. Since $\tilde{V}$ is concave on $[c_*, X]$, and $\tilde{V}(c) \leq \eta(c)$ on $[\theta, X]$ we have that $\tilde{V}'(c_*) > 0$ and therefore

$$0 = \frac{1}{2}\sigma^2\tilde{V}''(c_*) = (\rho + \pi + \lambda)(\tilde{V}(c_*) - \eta(c_*)) - (rc_* + \mu)\tilde{V}'(c_*) \ < \ 0,$$

which provides the required contradiction and thus proves the concavity of $\tilde{V}$ over the interval $[\theta, X]$. The assumption that $Y(c) \leq k + c$ implies that $\tilde{V}(X; X) = \alpha(X) - X$ diverges to $-\infty$ as X goes to infinity. Therefore, since $\tilde{V}(c; X)$ is non-decreasing by construction, we get that $\tilde{V}(0; X)$ diverges to $-\infty$ as X goes to infinity. Uniqueness of the threshold C^* will follow from the verification result below: since $V(c; C^*)$ is the value function, there can be only one C^*.

In order to show that $\zeta(C^*) = 0$, and thus complete the proof, suppose to the contrary. Since $\zeta(C^*)$ is the first cash level at which $\tilde{V}' = 0$, we have $\tilde{V}''(\zeta(C^*); C^*) \geq 0$ and therefore $\tilde{V}(\zeta(C^*); C^*) \geq \eta(\zeta(C^*))$ due to equation (70). On the other hand, since η is increasing and concave for $c \leq \zeta(C^*)$, we have

$$\ell(0) = \tilde{V}(0; C^*) = \tilde{V}(\zeta(C^*); C^*) \geq \eta(0).$$

However,

$$\eta(0) \ = \ \frac{\mu + \pi Y(0) + \lambda\tilde{V}(C^*; C^*)}{\rho + \pi + \lambda} = \frac{\mu + \pi Y(0) + \lambda\tilde{V}(\zeta(C^*); C^*)}{\rho + \pi + \lambda} > \ell(0) = \tilde{V}(\zeta(C^*); C^*)$$

because $\varphi < 1$ and $Y(0) \geq \ell(0)$ by assumption. This provides the required contradiction and completes the proof. ■

Lemma 40 *Under the hypotheses of Lemma 39, the function defined via*

$$V(c; Y(\cdot), \pi) = V(c \wedge C^*(Y(\cdot)); C^*(Y(\cdot))) + (c - C^*(Y(\cdot)))^+$$

is increasing, concave and satisfies the HJB equation

$$\max\{\mathcal{L}V(c) + \mathcal{F}V(c) + \pi(Y(c) - V(c)), 1 - V'(c), \ell(c) - V(c)\} = 0$$

and therefore coincides with the value function of the problem (68).

Proof. The claim follows directly from Lemma 39 by the same argument as in the proof of Lemma 16 and Proposition 1 as soon as we prove concavity. To do the latter, let $X = C^*$. As we have shown in the proof of Lemma 39, $\tilde{V}(c; X)$ is concave, increasing and satisfies $\tilde{V}(c; X) < \eta(c)$ for $c \in [\theta, X]$ where θ is the highest cash level below X at which $\tilde{V}(c; X) = \eta(c)$. It therefore remains to show that $\tilde{V}(c; X)$ is concave for $c \in [0, \theta]$.

Suppose on the contrary that $\tilde{V}$ is not concave on $[0, \theta]$ and let c_* be the first cash level from the right at which $\tilde{V}''(c_*) = 0$. Since $\tilde{V}'(X; X) = 0$ and $\tilde{V}$ is concave on $[c_*, X]$, we have $\tilde{V}'(c_*) > 0$. Furthermore, by Lemma 39, $\tilde{V}' > 0$ for all $c \geq 0$. Furthermore, as we have shown in the proof of Lemma 39, $\tilde{V}(0; X) < \eta(0)$. Therefore, by the concavity of η, the function $\tilde{V}$ cannot be convex on $[0, c_*]$. So, let $z < c_*$ be the first cash level to the left of c_* at which $\tilde{V}''(z) = 0$. Then, $\tilde{V}''(c) < 0$

for $c \in (c_*, \theta)$, and $\tilde{V}''(c) > 0$ for $c \in (z, c_*)$. Hence, $\tilde{V}'(c)$ has a positive local maximum at c_* and, consequently, $\tilde{V}'''(c_*) \leq 0$. Differentiating (70) at $c = c_*$, we get

$$(\rho + \lambda + \pi - r)\tilde{V}'(c_*) - (\rho + \lambda + \pi)\eta'(c_*) = \frac{\sigma^2}{2}\tilde{V}'''(c*) \leq 0.$$

Consequently, since $\tilde{V}'$ is increasing on $[z, c_*]$ and η' is decreasing on $[z, c_*]$, we get that

$$(\rho + \lambda + \pi - r)\tilde{V}'(c) - (\rho + \lambda + \pi)\tilde{\eta}'(c) \leq (\rho + \lambda + \pi - r)\tilde{V}'(c_*) - (\rho + \lambda + \pi)\eta'(c_*) \leq 0$$

for $c \in [z, c_*]$. Therefore, differentiating (70), we get

$$\frac{\sigma^2}{2}\tilde{V}'''(c) = -(rc + \mu)\tilde{V}''(c) + (\rho + \lambda + \pi - r)\tilde{V}'(c) - (\rho + \lambda + \pi)\eta'(c) \ < \ 0\,.$$

Thus, the function $\tilde{V}''$ is decreasing on $[z, c_*]$ and it follows that $0 = \tilde{V}''(z) > \tilde{V}''(c_*) = 0$, which is a contradiction.[9] ■

Lemma 41 *For any $\pi > 0$, the map $Y \ \to \ V(c; Y(\cdot), \pi)$ is a contraction in the* sup*-norm, with the contraction constant not greater than $\pi/(\rho + \pi)$.*

Proof. Pick Y_1, Y_2 and let $(C_{1t}, D_{1t}, f_{1t}, \tau_{01})$ and $(C_{2t}, D_{2t}, f_{2t}, \tau_{02})$ be the corresponding optimal policies and fix an initial cash level $c \geq 0$. Then,

$$\begin{aligned}
&V(c; Y_1(\cdot), \pi) \ - \ V(c; Y_2(\cdot), \pi) \\
&= \ E_c\left[\int_0^{\tau_Y \wedge \tau_{01}} e^{-\rho t}(dD_{1t} - f_{1t-}dN_t) \ + \ 1_{\tau_{10} < \tau_Y} e^{-\rho\tau_{01}}\ell(0) \ + \ 1_{\tau_Y < \tau_{01}} e^{-\rho\tau_Y} Y_1(C_{1\tau_Y})\right] \\
&- \ E_c\left[\int_0^{\tau_Y \wedge \tau_{02}} e^{-\rho t}(dD_{2t} - f_{2t-}dN_t) \ + \ 1_{\tau_{20} < \tau_Y} e^{-\rho\tau_{02}}\ell(0) \ + \ 1_{\tau_Y < \tau_{02}} e^{-\rho\tau_Y} Y_2(C_{2\tau_Y})\right] \\
&\geq \ E_c\left[\int_0^{\tau_Y \wedge \tau_{02}} e^{-\rho t}(dD_{2t} - f_{2t-}dN_t) \ + \ 1_{\tau_{20} < \tau_Y} e^{-\rho\tau_{02}}\ell(0) \ + \ 1_{\tau_Y < \tau_{02}} e^{-\rho\tau_Y} Y_1(C_{2\tau_Y})\right] \\
&- \ E_c\left[\int_0^{\tau_Y \wedge \tau_{02}} e^{-\rho t}(dD_{2t} - f_{2t-}dN_t) \ + \ 1_{\tau_{20} < \tau_Y} e^{-\rho\tau_{02}}\ell(0) \ + \ 1_{\tau_Y < \tau_{02}} e^{-\rho\tau_Y} Y_2(C_{2\tau_Y})\right] \\
&= \ E_c\left[1_{\tau_Y < \tau_{02}} e^{-\rho\tau_Y}(Y_1(C_{2\tau_Y}) - Y_2(C_{2\tau_Y}))\right] \\
&\leq \ \sup_{c \in \mathbb{R}_+} |Y_1(c) - Y_2(c)| E_c\left[e^{-\rho\tau_Y}\right] \ = \ \frac{\pi}{\rho + \pi} \sup_{c \in \mathbb{R}_+} |Y_1(c) - Y_2(c)|.
\end{aligned}$$

Interchanging Y_1, Y_2, we get the reverse inequality

$$V(c; Y_1(\cdot)) \ - \ V(c; Y_2(\cdot)) \ \geq \ -\frac{\pi}{\rho + \pi} \sup_{c \in \mathbb{R}_1} |Y_1(c) - Y_2(c)|\,,$$

and the claim follows. ■

[9] In the above argument, we assume that the zeros of the functions under consideration cannot accumulate. This follows from the fact that all functions involved are real analytic in c.

Consider the complete metric space $\mathcal{C}$ of pairs (V^1, V^2) of concave functions on $\mathbb{R}_+$ such that both $V^1(c) - c$ and $V^2(c) - c$ are bounded, equip $\mathcal{C}$ with the metric

$$\|(V^1, V^2) - (\tilde{V}^1, \tilde{V}^2)\| \;=\; \max\left\{ \sup_{c\in\mathbb{R}_+} |V^1(c) - \tilde{V}^1(c)|, \sup_{c\in\mathbb{R}_+} |V^2(c) - \tilde{V}^2(c)| \right\},$$

and define a mapping $F : C \to \mathcal{C}$ by setting

$$F(V^1, V^2) \;=\; (V(c; V^2(\cdot), \pi_{LH}) \;,\; V(c; V^1(\cdot), \pi_{HL})).$$

Then, the following is true:

Lemma 42 *The map F is a contraction in $\mathcal{C}$ and therefore has a unique fixed point that can be found by successful iterations. This fixed point is the unique solution to the system (36) and the corresponding policy is the optimal policy. The corresponding boundaries C_L^* and C_H^* are the optimal cash holdings.*

Proof of Lemma 42. We only need to show the contraction property. To this end fix an arbitrary quadruple of functions (V^1, V^2), $(\tilde{V}^1, \tilde{V}^2) \in \mathcal{C} \times \mathcal{C}$ and observe that by direct application of the result of Lemma 41 we have

$$\begin{aligned}
&\|F(V^1, V^2) - F(\tilde{V}^1, \tilde{V}^2)\| \\
&= \max\left\{ \sup_{c\in\mathbb{R}_+} |V(c; V^2(\cdot), \pi_{LH}) - V(c; \tilde{V}^2(\cdot), \pi_{LH})|, \sup_{c\in\mathbb{R}_+} |V(c; V^1(\cdot), \pi_{HL}) - V(c; \tilde{V}^1(\cdot), \pi_{HL})| \right\} \\
&\leq \max\left\{ \frac{\pi_{LH}}{\rho + \pi_{LH}} \sup_{c\in\mathbb{R}_+} |V^2(c) - \tilde{V}^2(c)|, \frac{\pi_{HL}}{\rho + \pi_{HL}} \sup_{c\in\mathbb{R}_+} |V^1(c) - \tilde{V}^1(c)| \right\} \\
&\leq \frac{\max\{\pi_{LH}, \pi_{HL}\}}{\rho + \max\{\pi_{LH}, \pi_{HL}\}} \|(V^1, V^2) - (\tilde{V}^1, \tilde{V}^2)\|,
\end{aligned}$$

which is what had to be proved. ■