Observable Long-Run Ambiguity and Long-Run Risk

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Abstract

I propose a general equilibrium model with spanned and unspanned observable macro variables. Spanned long-run GDP and inflation risk explains variations in the yield curve. Unspanned model uncertainty premiums for both long-run risk components drive bond premiums and bond volatility. The conditional volatility of consumption and inflation is constant, but accounting for uncertainty premiums generates a skewed option implied volatility smile.

Keywords: Bond premium, unspanned volatility, unspanned expected bond return, interest rate option, implied volatility, interest rates, yield curve

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1. Introduction

This paper introduces a general equilibrium interest rate and interest rate option pricing model, in which two observable, but unspanned, macro uncertainty premiums drive expected excess bond returns, bond volatility and the skewed option implied volatility smile.

Empirical research has documented that bond volatilities and bond premiums are unspanned by the yield curve. Collin-Dufresne and Goldstein (2002) use at-the-money straddles and swap rates to show that interest rate volatility is not spanned by interest rates.[2] Cochrane and Piazzesi (2005) find a single unspanned factor that predicts bond premiums and that is unrelated to level, slope and curvature.[3] Duffee (2010) finds that the fifth principal component of the Treasury yield curve (a hidden factor) significantly predicts bond premiums. [4] Ludvigson and Ng (2009) shows that the unspanned bond premium is driven by macroeconomic fundamentals.

I provide a parsimonious asset pricing model that provides an economic story to these findings. My endowment economy has a homoscedastic consumption and inflation process. In real-time, the investor observes their conditional expected growth rates, which themselves follow homoscedastic processes. These expected growth rates determine the long-run behavior of the economic system. I therefore call them long-run risk components for GDP growth and for inflation, respectively.[5]

[2]Heidari and Wu (2003) confirm the existence of an unspanned volatility factor in the yield curve.

[3]Litterman and Scheinkman (1991) find that the first three principal components of the Treasury yield curve (level, slope and curvature) explain more than 95% of yield curve variations. Dai and Singleton (2000), Duffie and Singleton (1997), Cochrane and Piazzesi (2010), and others, support this.

[4]In the data, the unspanned uncertainty premium for inflation explains 12% of the fifth principal component.

[5]This notation builds a connection to the unobserved expected growth rates in consumption and inflation that are used in the long-run risk literature. Compare Bansal and Yaron (2004), Bansal et al. (2007), Bansal and Shaliastovich (2006), Bansal and Shaliastovich (2009), Bansal and Shaliastovich (2010b), Piazzesi and Schneider (2006), Piazzesi and Schneider (2010), Drechsler (2009), Drechsler and Yaron (2010) among others. The results of my model do not rely on Epstein and Zin (1989), Epstein and Duffie (1992) type

The model relaxes the assumption that the investor knows the single correct model that describes both long-run risk components.[6] Real-time data from the Survey of Professional Forecasters (SPF) confirms that different econometric professionals from Wall Street apply different models for forecasting next quarter GDP growth and next quarter inflation. I use the cross-sectional standard deviation among these quarterly forecasts to construct a set of potentially correct long-run risk models.[7] I assume this time-varying set follows a heteroscedastic diffusion, which makes bond yields and bond premia heteroscedastic, as well.[8]

Although, the model is simplified and relies only on observable variables, the equilibrium insights are of importance. First, an ambiguity averse investor requires a model uncertainty premium if forecasts on the fundamental processes are disperse. One premium is paid for ambiguity about long-run GDP growth, while the other is paid for ambiguity about long-run inflation. Second, nominal bond yields load on the long-run risk factors and on their uncertainty premiums. Third, a variance decomposition, evaluated at the maximum likelihood estimates, reveals that the uncertainty premiums add virtually nothing to the time-variation of nominal bond yields. An econometrican looks at the model implied data and concludes that the yield curve does not span the uncertainty premiums.[9] Fourth, the equilibrium uncertainty premiums give rise to counter cyclical bond premiums. This equilibrium mechanism constitutes an alternative to the stochastic risk aversion

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preferences.

[6]The concept of model uncertainty and model ambiguity follows Knight (1921), Hansen and Sargent (2008), Chen and Epstein (2002) and is different to the concept of heteroscedasticity in fundamentals (Bekaert et al. (2009)).

[7]Patton and Timmermann (2010) analyze the term structure of survey based macro forecasts and concludes that the economic reason for observing disperse forecasts is model disagreement. My model approach accounts for this. Anderson et al. (2009), Ulrich (2010) use similar assumptions.

[8]Shaliastovich (2009) and Bansal and Shaliastovich (2010a) use the cross-sectional standard deviation in SPF forecasts as a measure for confidence risk and assume it follows a jump-diffusion process.

[9]The concept of unspanned factors, following the tradition in Collin-Dufresne and Goldstein (2002) and Cochrane and Piazzesi (2005) is based on variance decompositions and picking up of correlations.

framework in Campbell and Cochrane (1999).[10] Fifth, my measure for the uncertainty premium explains 12% of the data implied fifth principal component of the yield panel. This provides a macroeconomic interpretation to the hidden factor of Duffee (2010).

Sixth, variations in the uncertainty premiums drive variations in the snake-shaped volatility of yield changes.[11] The model implied volatility smile of interest rate options is skewed and entirely driven by the unspanned uncertainty premiums. The model predicts that an increase in ambiguity about long-run inflation increases the level of the smile and amplifies the skew. On the contrary, an increase in uncertainty about the long-run GDP growth prospects lowers the smile and flattens the skew. This counter intuitive behavior for the latter occurs because bond prices rise in times of increased GDP uncertainty, while they fall in times of rising inflation uncertainty. The intuition for this result is that real and nominal bonds are more valuable in times of increased GDP uncertainty, because these assets are recession hedges. In contrast, the value of nominal bonds is less valuable in times of higher inflation uncertainty.[12]

The paper is structured as follows. Section 2 describes the model and derives equilibrium bond yields, bond premia, yield volatility and specifies the interest rate option contract of interest. I estimate the model in section 3 with bond yield and macro data. Results are explained and implications for bond premia and bond volatility are visualized in section 3, as well. Section 4 quantifies the detection error. Section 5 concludes. The appendix contains derivations and proofs.

[10] There is a growing literature on the importance of model uncertainty in asset pricing. Recent examples are Cagetti et al. (2002), Gagliardini et al. (2009), Kleshecheslski and Vincent (2009), Ulrich (2010), Anderson et al. (2003), Hansen and Sargent (2008), Hansen et al. (2005), Maenhout (2004), Liu et al. (2005) and Maenhout (2006), among others.

[11] Piazzesi (2005) and Piazzesi (2003) discovered the snake pattern.

[12] Ulrich (2010) shows that log utility together with inflation ambiguity can recover the positive term premium in U.S. Treasury bonds. Liu et al. (2005) and Drechsler (2009) show that model uncertainty about the jump component in the endowment growth rate can account for the skewed volatility smile of equity options.

2. Model

I work with an endowment economy. Time is continuous and varies over $t \in [0, ..., \infty)$. I assume a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q^0)$, where Q^0 stands for the reference macro model. I denote expectations under Q^0 as $E[.]$ instead of $E^{Q^0}[.]$. I endogenously determine the robust probability measure and denoted it Q^h. All Brownian motions are pairwise orthogonal.

2.1. Assumptions on the Benchmark Economy

Realized growth of the investor's endowment follows a homoscedastic process with a time-varying trend growth rate z

$$d \ln c_t = (c_0 + z_t)dt + \sigma_c dW_t^c, \tag{1}$$

with $c_0 > 0$ and $\sigma_c > 0$. The mean zero trend growth rate, z, follows a continuous-time AR(1) process

$$dz = \kappa_z z dt + \sigma_{1z} dW^r + \sigma_{2z} dW^w \tag{2}$$

with $\kappa_z < 0$, $\sigma_{1z} > 0$, and $\sigma_{2z} < 0$. I call z the realization of the long-run consumption risk process (or trend consumption growth).

The exogenous process for inflation, $d \ln p$, follows also a homoscedastic process

$$d \ln p_t = (p_0 + w_t)dt + \sigma_p dW_t^p, \tag{3}$$

with $p_0 > 0$ and $\sigma_p > 0$. The mean zero trend growth rate of inflation, w, follows a continuous-time AR(1) process

$$dw = \kappa_w w dt + \sigma_w dW^w, \tag{4}$$

with $\kappa_w < 0$ and $\sigma_w > 0$. I call w the long-run inflation risk component (or trend inflation). I account for the negative correlation between z and w through $\sigma_{2z} < 0$ (Piazzesi and Schneider (2006) and Ulrich (2010)). The realizations of z and w are observed in real-time.

Instead of assuming Epstein and Zin (1989) preferences, as done in the long-run risk literature (Bansal and Yaron (2004), Piazzesi and Schneider

(2006)), I assume the investor has simple logarithmic utility. His expected life-time utility in period t is $E_t\left[\int_t^{\infty} e^{-\rho s} \ln c_s ds\right]$, where $\rho > 0$ is the subjective time discount factor.[13]

2.2. Assumptions on the Robust Economy

The investor is uncertain about which transition density describes the empirical dynamics of w and z best. This means that the investor faces ambiguity about the long-run risk model. The investor believes that equations (2) and (4) are a good approximation for the unknown data generating process of the long-run risk components. They describe his benchmark model (reference model).

The investor observes in each period t a set of potentially correct long-run risk models. Different long-run risk models differ in their conditionally expected growth rates. After observing z_t and w_t, the investor uses likelihood ratio tests to quantify the accuracy of his trusted benchmark model in comparison to the other models from the set. Due to the stochastic nature of z and w, the investor is exposed to time-varying likelihood ratios.

I define an observed change in the log-likelihood ratio as $d \ln \frac{dQ_t^h}{dQ_t^0}$. An increase in the latter is bad news for the trustworthiness of the benchmark model, because it implies that the observed realization of dz_t and dw_t has probably been generated by the worst-case model Q^h and not by the benchmark model Q^0. While in general, the investor could focus on every potentially correct model, the min-max investor focuses only on the benchmark model and on the worst-case model.

The Gaussian processes z and w imply that the expected instantaneous change of the log-likelihood ratio, under the worst-case measure, is given by

$$E_t^h\left[d \ln \frac{dQ_t^h}{dQ_t^0}\right] = \frac{1}{2}(h_t^r)^2 dt + \frac{1}{2}(h_t^w)^2 dt \tag{5}$$

, where h_t^r are instantaneous drift distortions to long-run consumption risk, while h_t^w are instantaneous drift distortions to long-run inflation risk. Mathematically this implies $dW_t^{r,h} + h_t^r dt = dW_t^r$ and $dW_t^{w,h} + h_t^w dt = dW_t^w$, where

[13] The model implications hold for more general utility functions. It is an advantage of log utility that it supports closed form solutions.

dW^h is a Brownian motion under the robust belief Q^h.

I constrain the growth rate of relative entropy between different long-run risk models in the set, by constraining each of the two components separately. I assume that the expected relative entropy component that arises because of long-run consumption ambiguity is constrained by

$$\frac{1}{2}(h_t^r)^2 dt \leq A^r(\eta_t^r)^2 dt, \tag{6}$$

with $A^r > 0$ and η^r follows a CIR process. The last equation says that the empirical trustworthiness of the benchmark long-run consumption risk model is time-varying over the business cycle. For the special case of $A^r = 0$, the investor has 100% confidence in the benchmark model of equation (2). For all other cases, the investor faces a time-varying amount of long-run consumption uncertainty. I assume the time-variation follows

$$d\eta_t^r = (a_{\eta^r} + \kappa_{\eta^r}\eta_t^r)dt + \sigma_{\eta^r}\sqrt{\eta_t^r}dW_t^{\eta^r}, \tag{7}$$

with $a_{\eta^r > 0}$, $\kappa_{\eta^r} < 0$ and $\sigma_{\eta^r} > 0$.

Analogously, I assume that the expected growth in relative entropy, which arises because of long-run inflation ambiguity, is constrained by

$$\frac{1}{2}(h_t^w)^2 dt \leq A^w(\eta_t^w)^2 dt, \tag{8}$$

where $A_w > 0$. I assume the time-varying amount of inflation uncertainty follows a CIR process

$$d\eta_t^w = (a_{\eta^w} + \kappa_{\eta^w}\eta_t^w)dt + \sigma_{\eta^w}\sqrt{\eta_t^w}dW_t^{\eta^w} \tag{9}$$

with $a_{\eta^w} > 0$, $\kappa_{\eta^w} < 0$ and $\sigma_{\eta^w} > 0$.

To summarize: The investor is uncertain about the transition density of the long-run consumption and long-run inflation risk model

$$dz = \kappa_z z dt + +\sigma_{1z}(dW^{r,h} + h^r dt) + \sigma_{2z}(dW^{w,h} + h^w dt) \tag{10}$$

$$dw = \kappa_w w dt + \sigma_w(dW^{w,h} + h^w dt). \tag{11}$$

The amount of the worst-case distortions are indirectly constrained through observed expected growth rates of relative entropy between the worst-case and the benchmark long-run risk model.

2.3. Equilibrium: Deciding which Model to Use

The investor has min-max preferences. In order to find the worst-case long-run risk model, the investor solves

$$\min_{Z \in Z(\eta^r, \eta^w)} E^Z \left[\int_t^\infty e^{-\rho(s-t)} \ln c_s ds | \mathcal{F}_t \right] \tag{12}$$

$$s.t. (1), (10), (5), (8), (6). \tag{13}$$

The solution to the minimization problem follows along the lines of Ulrich (2010) and is summarized in the following proposition.

Proposition 1 *The investor distorts the benchmark long-run risk dynamics by a time-varying amount* h^r and h^w

$$h^r(t) = m^r \eta^r(t), m^r := -\sqrt{2A^r} \in \mathcal{R}^- \tag{14}$$

$$h^w(t) = m^w \eta^w(t), m^w := \sqrt{2A^w} \in \mathcal{R}^+. \tag{15}$$

The appendix contains the derivation.

Under the benchmark model, the economy is homoscedastic. Risk premiums and asset return volatilities are constant. The robust economy is heteroscedastic with time-varying uncertainty premiums and time varying asset return volatilities. The time-variation of these financial variables is exclusively driven by the observed amount of macro ambiguity.

2.4. Equilibrium: Marginal Rate of Substitution

I collect h^r and h^w into a vector h, i.e. $h_t = (h_t^r \, h_t^w)'$. Similarly, I collect the ambiguous shocks dW^r and dW^w into a vector dW^z, i.e. $dW_t^w = (dW_t^r \, dW_t^w)'$. In period T, the likelihood ratio between the worst-case and the benchmark long-run risk model is $\frac{dQ_T^h}{dQ_T^0}$. I call this likelihood ratio a_T. The Gaussian dynamics of z and w imply

$$a_T := \frac{dQ_T^h}{dQ_T^0} = \exp\left(-\frac{1}{2} \int_0^T h_t' h_t dt + \int_0^T h_t \cdot dW_t^z \right), a_0 \equiv 1. \tag{16}$$

The last equation stands for the amount of model uncertainty. In technical terms, this coincides with the amount of entropy between the worst-case and the benchmark model. The relative change of realized uncertainty is a martingale under the benchmark model

$$\frac{da_t}{a_t} = h_t \cdot dW_t^z. \tag{17}$$

This implies that the investor does not expect to learn which model is correct, i.e. $E_t\left[\frac{da_t}{a_t}\right] = 0$.

The intertemporal marginal rate of substitution (MRS) contains information on the degree of consumption risk and the degree of model uncertainty. I define the MRS as m. It is endogenous in my economy, i.e.

$$m_{t,t+\Delta} = e^{-\rho\Delta}\left(\frac{c_{t+\Delta}}{c_t}\right)^{-1}\frac{a_{t+\Delta}}{a_t}. \tag{18}$$

The evolution of the MRS reveals the equilibrium real interest rate and the equilibrium market prices of consumption risk and model uncertainty:

$$-\frac{dm_t}{m_t} = \frac{dc_t}{c_t} - \frac{da_t}{a_t}. \tag{19}$$

According to assumption (1), the market price of consumption risk is σ_c. It is constant and paid for shocks to realized consumption. The endogenous dynamic in (17) reveals that the market price for ambiguity about the long-run risk dynamics is: $-h_t^r \in \mathcal{R}^+, \forall t \geq 0$ and $-h_t^w \in \mathcal{R}-, \forall t \geq 0$. The first is the market price for uncertainty about the long-run consumption growth dynamic, while the latter is the market price for uncertainty about the long-run inflation dynamic. The real interest rate coincides with $r_t = \rho + c_0 - \frac{1}{2}\sigma_c^2 + z_t$.

The nominal equilibrium SDF is $m^\$$

$$m_{t,t+\Delta}^\$ = m_{t,t+\Delta}\frac{p_t}{p_{t+\Delta}}. \tag{20}$$

The equilibrium nominal short rate equals $R_t := -E_t[\frac{dm_{0,t}^\$}{m_{0,t}^\$}] = r_t + w_0 - \frac{1}{2}\sigma_p^2 + w_t$.

The real short, and the nominal short-rates do not depend on macro uncertainty. Interest rates on longer duration Treasury bonds depend on macro uncertainty, because these long-term interest rates are risk and uncertainty adjusted expectations of future short rates. A min-max investor prices assets under the endogenously determined worst-case scenario.

2.5. Equilibrium: Bond Market

Consumption, inflation, the real and nominal short rate, the market price of risk are homoscedastic. Nevertheless, yields in the economy are heteroscedastic, because the market price of uncertainty is heteroscedastic.

$B_t(\tau)$ is the price at time t of a τ maturity real bond. Its price is exponentially affine in the long-run consumption risk factor and in the uncertainty premiums:

$$B_t(\tau) = e^{A^r(\tau)+B^r(\tau)S_t, \quad S\equiv(z\, h^w\, h^r)'}. \tag{21}$$

The continuously compounded real interest rate is affine in the long-run consumption risk factor and its uncertainty premiums

$$y_t^r(\tau) = a^r(\tau) + b^r(\tau)S_t, \tag{22}$$

with $a^r(\tau) := -A^r(\tau)/\tau$ and $b^r(\tau) := -B^r(\tau)/\tau$. The loadings are deterministic functions of the parameters of the economy. The appendix contains details on the derivation.

I denote the price at time t of a τ maturity nominal Treasury bond as $N_t(\tau)$. Its equilibrium price is exponentially affine in long-run consumption risk, long-run inflation risk, and the market prices of long-run ambiguity:

$$N_t(\tau) = e^{A^n(\tau)+B^n(\tau)X_t, \quad X\equiv(w\, S)'}. \tag{23}$$

with $a^n(\tau) := -A^n(\tau)/\tau$ and $b^n(\tau) := -B^n(\tau)/\tau$. Nominal interest rates are affine in the risk and uncertainty factors

$$y_t^n(\tau) = a^n(\tau) + b^n(\tau)X_t, \tag{24}$$

with $a^n(\tau) := -A^n(\tau)/\tau$ and $b^n(\tau) := -B^n(\tau)/\tau$. The appendix specifies the factor loadings as functions of the underlying economy.

The slope of the nominal yield curve, measured as the 40-quarter nominal yield minus the nominal short-rate, is an affine function in expected inflation and expected consumption growth and in their corresponding model uncertainty premium:

$$y^n(40) - R = (b_w^n(\tau) - 1)w_t + (b_z^n(40) - 1)z_t + b_{h^r}^n(40)h_t^r + b_{h^w}^n h_t^w. \tag{25}$$

The instantaneous bond premium of a nominal Treasury bond varies with the degree of long-run ambiguity

$$E_t^h\left[\frac{dN_t(\tau)}{N_t(\tau)} - R_t dt\right] = (\sigma_{2z}B_z^n(\tau) + \sigma_w B_w^n(\tau))(-h_w(t))dt+ \\ + \sigma_{1z}B_z^n(\tau)(-h^r(t))dt. \quad (26)$$

The bond price loadings are negative. This implies that an expected increase in the cross-sectional dispersion of inflation forecasts, increases $h^w > 0$, which leads to an increase in the bond premium across all maturities. On the other hand, an expected increase in the cross-sectional dispersion of consumption forecasts, lowers $h^r < 0$, which reduces the bond premium across all maturities.

The type of uncertainty matters for the bond premium. While bonds hedge consumption uncertainty, they do not hedge inflation uncertainty. This results in a positive inflation uncertainty premium and a negative consumption uncertainty premium. The bond premium can have different signs, depending on the relative magnitude of long-run consumption uncertainty versus long-run inflation uncertainty.

The volatility of interest rate changes is time-varying as well. If the macro economy becomes more uncertain, real and nominal bond yields become more volatile. The quadratic variation of changes in yields depends on the heteroscedastic uncertainty premiums. Mathematically this means

$$< dy_t^n(\tau), dy_t^n(\tau) > = (b_{h^r}^n(\tau))^2 m^r \sigma_{\eta^r} h_t^r dt + (b_{h^w}^n(\tau))^2 m^w \sigma_{\eta^w}^2 h_t^w dt \quad (27)$$

$$< dy_t^r(\tau), dy_t^r(\tau) > = (b_{h^r}^r(\tau))^2 m^r \sigma_{\eta^r} h_t^r dt + (b_{h^w}^r(\tau))^2 m^w \sigma_{\eta^w}^2 h_t^w dt \quad (28)$$

where b_{h^r} and b_{h^w} are the corresponding yield loadings for the market price of consumption uncertainty and the market price of inflation uncertainty.

2.6. Equilibrium: Option Market

Bond options carry a time-varying implied volatility. Changes in the volatility of options are induced by changes in the trustworthiness of the benchmark long-run risk dynamics. This implies that periods of increased macro uncertainty lead to endogenous fluctuations in the option implied volatilities.

While the precise type of bond option does not matter, I consider an interest rate option that is traded on the CBOE. Let C be a floor on an interest rate $y^n(\tau)$. Being long a floor entitles the owner of the option to receive at maturity T the maximum of $(y_T(\tau) - K, 0)$, where K is a fixed interest rate level. The value of this option depends on the maturity of the contract, strike, current value of the yield of interest, current nominal short rate. I fix the notional of the contract to \$100. Its equilibrium price is

$$C(t, T, K, y_t(\tau), R_t) := \$100\, E\left(m^{\$}_{t,T}(y_T(\tau) - K)^+\right). \tag{29}$$

Under rational expectations (benchmark model), the option price follows the Black (1976) model. The implied volatility would be constant and the model would fail to reproduce the skewed option smile as observed in the data. Accounting for model uncertainty (worst-case model), makes option prices follow a Black (1993) type model. Variations in implied volatility coincide with variations in macro uncertainty. The model can account for smiles and skews in the option market.[14] The casual interpretation of implied volatility as a fear index (uncertainty index) is correct in a model with uncertainty about the long-run risk dynamics.

In the empirical section, I determine the price of a call option via Monte Carlo simulation. I use an Euler-Marujuama discretization scheme, and simulate the system on a daily interval. I plug the resulting call price into Black (1976) formula to invert the option implied volatility.

[14]Recent equilibrium models have exclusively focused on equity options. Liu et al. (2005) and Drechsler (2009) show that ambiguity about rare events can explain the skewed volatility smile in equity options. Buraschi et al. (2009), David and Veronesi (2009), Buraschi and Jiltsov (2006), David and Veronesi (2002) show that learning about the fundamental processes in the economy helps to explain why dispersion in forecasts explain equity option prices. Drechsler and Yaron (2010), Bollerslev et al. (2009b), Eraker and Shaliastovich (2008), Bollerslev et al. (2009a), Shaliastovich (2009) show that stochastic volatility in consumption growth together with Epstein-Zin preferences can explain equity option prices.

3. Empirical Part

The model is estimated with macro, bond yield, and bond variance data. I analyze the model's implications for expected bond returns, and the smile and skew of option implied volatilities.

3.1. Data

The data is from 1972 to 2009. The data frequency is quarterly. I use the following data to match the exogenous processes of the economy. First, $d \ln c$ is matched with realized GDP growth. Second, $d \ln p$ is matched with realized inflation. Third, z_t is matched with the demeaned median forecast in period t, of next quarter GDP growth. The forecast is taken from the Survey of Professional Forecasters (SPF). Fourth, w_t is matched with the demeaned median SPF forecast in period t, of next quarter inflation. Fifth, η_t^w coincides with the cross-sectional standard deviation in t across one quarter ahead SPF inflation forecasts. Sixth, η_t^r is matched with the cross-sectional standard deviation in t across one quarter ahead SPF GDP growth forecasts. I use GDP data instead of consumption data, because there is no forecast and model disagreement data for consumption growth available. To remove seasonality in η_t^w and η_t^z, I use a 4-quarter moving average, which implies that I use the $t-3, t-2, t-1, t$ dispersion to construct the non-seasonal t-measure for η^w and η^r. The appendix contains a more detailed description of the macro data.

Patton and Timmermann (2010) econometrically analyze the economic reason for disagreement in macro forecasts. Their analysis shows that disagreement in models is a plausible economic reason for disperse forecasts. First, the longer the forecast horizon, the higher the dispersion in forecasts. Second, dispersion in forecasts persist over time. Anderson et al. (2009) and Ulrich (2010) also use the cross-sectional dispersion in forecasts as a measure for the amount of macro ambiguity.[15]

I match the model output with the following financial data. First, ten panels of continuously compounded nominal Treasury bond yields of maturity one year to ten years. Second, six panels of continuously compounded

[15] Buraschi et al. (2009), David and Veronesi (2009), Buraschi and Jiltsov (2006), David and Veronesi (2002), among others, use dispersion in forecasts as a measure for heterogeneity in beliefs.

real Treasury bond yields of maturity five years to ten years. Real bond yields coincide with yields of Treasury Inflation Protected Securities (TIPS). Third, ten panels of variances of continuously compounded nominal Treasury yields with maturity of one year to ten years. The quarterly variance within a quarter is estimated as the sum of quadratic daily yield changes within the corresponding quarter. The appendix contains details on the exact construction of the financial data.

A first look at macro data reveals that z and w are unbiased predictors of realized GDP growth and realized inflation, respectively. The former explains 9% of the variance of realized GDP growth, while the latter explains 65% of the variance of realized inflation.

A first look at the bond data reveals interesting relations between the macro and the bond market. I denote the principal components of the panel of nominal yields as PC. PC1 stands for the first principal component, PC2 stands for the second, and so on. First, variations in $X_t = (w_t \, z_t \, \eta_t^r \, \eta_t^w)$ explain 63% of PC1 variations, with w_t and η_t^w having t-stats bigger than 2. Second, variations in X explain variations in PC2, with w_t and η_t^r having t-stats bigger than 2. Third, variations in X explain 19% of PC3 variations, with w_t and η_t^w having t-stats bigger than 2. Fourth, variations in X explain 4% of PC4 variations, while z_t has a t-stat of bigger than 2. Fifth, 12% of variations in PC5 are explained by variations in η^w. Sixth, 23% of variations in the slope (40-quarter yield minus federal funds rate) are explained by w_t and η_t^r. The negative loading on these factors is consistent with the model implied sign in equation (25).

The data evidence suggests that uncertainty on trend inflation significantly explains variations in PC1, PC3, and PC5. It further implies that uncertainty about trend GDP growth explains variations in PC2. Cochrane and Piazzesi (2005) find that a single factor predicts bond returns. This factor is mostly unrelated to the first three principal components of nominal yields. Duffee (2010) finds that the fifth principal component is a hidden factor in the yield curve which has substantial predictive power for bond returns. My first look at the data reveals that cross-sectional dispersion in inflation forecasts, which is part of the bond premium in my model, explains

12% of variations in the fifth principal component.

A first look at the volatility of yield changes reveals the following. First, one factor explains 94% of bond yield variance fluctuations. Second, regressing this factor on $(\eta^w)^2$ and $(\eta^r)^2$ reveals that both factors together explain 20% of its variation. The t-stat of both uncertainty factors is around -2.4.

3.2. Econometric Method

I use 29 measurement equations and 4 observable state variables in the estimation. I use 10 time-series of nominal yields, 10 time-series of nominal yield variances, one time-series of the federal funds rate, six time-series of real yields and two time-series of macro growth (inflation and GDP growth). The four state equations are trend GDP growth, trend inflation, cross-sectional dispersion in inflation forecasts and cross-sectional dispersion in GDP growth forecasts.

The estimation uses a conservative approach. I first, run a QML estimation with the observable state equations and inflation and GDP growth only in order to determine the 99% confidence interval of macroeconomic parameter estimates. Second, I run a one step QML with bond and macro data at the same time, with the constraint that the macro parameters lie within that 99% confidence interval. After the estimation, I check the model implications for the mean of the yield curve, its volatility, as well as the option volatility smile.

3.3. Empirical Findings:

3.3.1. Yield Curve

Figure 1 shows that the model explains term structure of nominal yields well. Panel A of Table 3 shows that changes in long-run inflation risk explains nearly all variations in yields. More specifically, 73.6% of variations in the 4-quarter yield arise from variations in long-run inflation risk. The remaining 26.4% arise from variations in long-run GDP risk. On the other hand, 97% of variations in the 40-quarter yield are due to variations on long-run inflation risk and the remaining 3% are due to variations in long-run GDP risk. The uncertainty premiums h^r and h^w are practically unspanned by the yield curve. The premiums affect the bond premiums and the option volatilities, but they do not explain variations in bond yields. The model basically endogenizes two unspanned macro factors that drive bond returns and bond volatilities. This provides a potential equilibrium explanation to the challenging empirical findings of Collin-Dufresne and Goldstein (2002), Joslin et al. (2009), Cochrane and Piazzesi (2005), Duffee (2010), and Ludvigson and Ng (2009) who find that factors that are unspanned by the yield curve drive bond returns and bond volatilities.

Figure 2 shows impulse responses of the yield curve to a one percent increase in the cross-sectional dispersion of both long-run risk factors. Confirming the results from Table 1, the impulse responses show that the unspanned uncertainty premiums affect the yield curve only marginally. Moreover, it matters for the yield curve, whether there is a shock to long-run GDP uncertainty or to long-run inflation uncertainty. A one percent increase in the cross-sectional dispersion of long-run GDP growth leaves the nominal short rate unaffected and lowers the 40-quarter yield by 0.3 basis points. The slope flattens by 0.3 basis points. Long-term bond yields fall as a result of rising bond prices. The equilibrium supports rising bond prices because an increase in GDP uncertainty is potentially good news for the future value of a bond, because bonds are recession hedges. The implications are different for inflation uncertainty. A one percent increase in the cross-sectional dispersion of long-run inflation does not affect the short rate but increases the 40-quarter yield by 9 basis points. The slope increases if the cross-sectional dispersion of inflation increases.

3.3.2. Bond Premium

Figure 3 compares empirical 4-quarter holding period returns of different bonds over the federal funds rate with the model counterpart. The empirical bond panel allows the exact measurement of 4 quarter holding period returns for all bond maturities. I construct excess returns by subtracting the federal funds rate.[16] As the model counterpart, I use the integrated four quarter expected excess return. Its unconditional correlation with the instantaneous

[16] I use the federal funds rate as the risk-free rate and not the four-quarter yield. This is the closest counterpart to the model implied expected excess return.

expected excess return from equation (26) is higher than 98%.[17]

The figure shows that the model matches the data counterpart. In the data, the term structure of 4-quarter holding period returns is upward sloping, from slightly above zero (4-quarter bond) to 2.2% for the 40 quarter bond. The term structure of excess 4-quarter holding period returns is volatile in the data. The population expected excess return for a 40-quarter bond lies within 0 and 4.5%. The model implied expected excess return lies in the upper part of the empirical confidence interval. It is entirely driven by unspanned uncertainty premiums and especially by the inflation uncertainty component, which in the data explains 12% of fifth principal component.

In order to analyze the 4-quarter holding period excess bond returns, I normalize the model and data implied ones to have zero mean and unit variance. The upper panel of Figure 4 shows the time-variation of both normalized excess returns for the ten year nominal bond. A predictive regression on a quarterly frequency shows that the model implied bond premium has predictive power. For the period of the Great Moderation, the R^2 equals 5%, the regression loading is 0.2 with a t-stat of 2.1.

The lower panel of Figure 4 shows that both unspanned uncertainty premiums affect the bond premium differently. The expected bond premium for long-run inflation ambiguity goes up at the beginning and during NBER recessions. The long-run GDP ambiguity premium behaves in an opposite way. The unconditional correlation with the long-run inflation ambiguity is -73%. The negative long-run GDP ambiguity premium leads to rising bond prices at the beginning and during recessions. In the model, both uncertainty

[17] The integrated four quarter expected excess return is

$$\begin{aligned}
&\int_t^{t+4} d\left(E_t^h\left[\frac{dN_t(\tau)}{N_t(\tau)} - R_t dt\right]\right) \\
&= \left(\sigma_{2z} B_z^n(\tau) + \sigma_w B_w^n(\tau)\right)\left(\left(\frac{m_w a_{\eta^w}}{\kappa_{\eta^w}^2} - \frac{h_t^w}{\kappa_{\eta^w}}\right)\left(e^{\kappa_{\eta^w}\cdot 4} - 1\right) + 4\frac{m_w a_{\eta^w}}{\kappa_{\eta^w}}\right) \\
&+ \sigma_{1z} B_z^n(\tau)\left(4\frac{m_r a_{\eta^r}}{\kappa_{\eta^r}} + \left(\frac{m_r a_{\eta^r}}{\kappa_{\eta^r}^2} - \frac{h_t^r}{\kappa_{\eta^r}}\right)\left(e^{\kappa_{4\eta^r}} - 1\right)\right)
\end{aligned}$$

premiums counter balance each other. The premium can therefore be positive or negative. Evaluated at the estimated QML parameters, the long-run inflation ambiguity premium is the dominant component. Panel B of Figure 3 shows that more than 90 percent of variations in expected bond returns are driven by variations in the long-run inflation ambiguity premium. The remaining part is attributed to variations in the long-run GDP ambiguity premium.

Figure 5 depicts that both unspanned ambiguity premiums affect the bond premiums differently. A one percent increase in the cross-sectional dispersion of inflation forecasts makes the bond investor request a 1.5% higher bond premium for a 40-quarter bond. This 1.5 multiple indicates a quantitatively important mechanism. On the other hand, a one percent increase in the cross-sectional dispersion of GDP growth forecasts makes the investor willing to pay a 2.5 basis point premium for holding a 40-quarter nominal bond. Intuitively, the investor pays a GDP uncertainty premium, because if the benchmark long-run GDP risk model turns out to be too optimistic, this positively affects the value of real and nominal bonds. On the contrary and consistent with the analysis in Ulrich (2010), inflation uncertainty increases in periods where the real value of nominal bonds falls. The bond premium for inflation uncertainty is therefore positive. Figure 5 and Figure 2 together show that an increase in the amount of ambiguity about the long-run inflation component increases the slope of the yield curve, as well as the expected bond premium. The model is therefore able to explain the empirical phenomenon that a steepening of the yield curve coincides with a higher uncertainty premium.

3.3.3. Volatility

Figure 6 visualizes the snake shaped model implied term structure of volatility of yield changes (Piazzesi (2005)). The model implied snake pattern is more pronounced than in my data sample, but it lies within two standard deviations of the data. The model attributes the volatility in the bond market to variations in unspanned ambiguity premiums. The unspanned nature of bond volatility is documented in Collin-Dufresne and Goldstein (2002).

Figure 7 shows that a one percent increase in the cross-sectional dispersion of the long-run risk factors, leads to rising volatility in the bond market.

More specifically, a one percent increase in the cross-sectional dispersion of trend inflation increases the head of the snake by 15%. Such a higher multiple shows that a modest increase in inflation ambiguity can lead to dramatic changes in bond market volatility. A 40-quarter bond reacts still with a multiple of 2.5. The impact of an increase in GDP uncertainty is qualitatively the same, but quantitatively different. The magnitude is roughly a tenth of the inflation uncertainty counterpart.

Figure 8 shows that changes in macro uncertainty changes the level and slope of the Black implied option volatilities. I perform a Monte Carlo simulation to determine the price in equation (29) on a three month call on nominal Treasury yields of maturity in 8, 12, 16 and 20 quarters, for different strikes. The x-axis denotes the moneyness, defined as $\ln \frac{K}{y_t^n(\tau)}$. Under the benchmark model, option implied volatilities are constant for all strikes. This is the case because the benchmark model is homoscedastic. The bond option price follows the Black (1976) model. Adding model uncertainty makes bond options follow a multi-dimensional Black (1993) type model. Unspanned macro uncertainty, in the form of unspanned premiums for the uncertain path of future economic growth (real and nominal) makes option implied volatilities have a skewed smile.

Collin-Dufresne and Goldstein (2002) provide evidence that option implied volatilities are driven by unspanned factors. My model is a theoretical explanation of how and why unspanned macro factors might affect option prices. My model implies that Black implied volatilities are higher the shorter the duration of the bond. The skew of implied volatilities turns to a smile the longer the duration of the bond. Statistical pricing models for interest rate options need either jumps in interest rates or a correlation between interest rates and its volatility (Jarrow et al. (2007) and Trolle and Schwartz (2000)). My analysis shows that a smooth macro equilibrium model with heteroscedastic uncertainty aversion is capable of generating a pronounced skew for options on short-duration yields. The skew is generated through the correlation of unspanned uncertainty and variations in bond yields. The model does not require a separate variance risk premium to generate the result.

Figure 9 summarizes the changes in implied volatility for a one percent increase in the cross-sectional standard deviation of long-run risk forecasts. This figure shows that unspanned ambiguity about the long-run risk components affects the skewed volatility smile differently. A one percent increase in the cross-sectional standard deviation of trend inflation forecasts increases the level of the smile and makes the skew more pronounced. This reflects that an increase in inflation ambiguity increases the fear in the bond market. In our model it is therefore consistent to interpret changes in implied volatility as a result of changes in economic fear.

On the contrary, a one percent increase in the cross-sectional standard deviation of trend GDP growth lowers the level of the volatility curve and lessens the skew of the smile. This implies that an increase in uncertainty does not automatically lead to higher fear in the bond market. The model provides a case where an increase in GDP uncertainty leads to falling implied volatility on the interest rate option market. The intuition for this behavior is that a more blurry outlook on GDP growth is good news for bond holders. The Euler equation shows that the value of bonds increases in times of a deteriorating GDP outlook.

All in one, it is the inflation uncertainty premium that dominates variations in the smile. Table 4 decomposes variations in the option implied smile into its two components. It shows that the inflation uncertainty premium drives roughly 99% of variations in implied volatilities.

4. Robustness

If the worst-case and the benchmark model are in a statistical sense far apart from each other, it becomes easy for an econometrician to tell which model generated the data. I therefore, determine the detection error probability (DEP), evaluated at the QML estimates. This probability denotes the likelihood, that a likelihood ratio test favors one model, although the data has been generated by the other model. Ulrich (2010) explains in detail how to derive DEPs in macro-finance models, and how they relate to Hansen and Sargent (2008).

The time-varying log-likelihood ratio between the worst-case model and the benchmark model is

$$\ln\left(\frac{dQ_T^h}{dQ_T^0}\right) = -\frac{1}{2}\int_0^T ((m^r)^2(\eta_t^r)^2 + (m^w)^2(\eta_t^w)^2)dt + \int_0^T (m^r\eta_t^r dW_t^r + m^w\eta_t^w dW_t^w). \tag{30}$$

The detection error probability depends on the market price of uncertainty and the realization of shocks to both long-run risk components. The appendix provides details on the derivation of the DEP.

The DEP, evaluated at the QML estimates is 23.5%. This says that after seeing the data, if the investor was to choose whether the data has been generated by the reference or the worst-case model, the likelihood ratio test would by statistical chance fool the investor in 23.5% of all cases.

5. Conclusion

We know from Campbell and Cochrane (1999) that stochastic risk aversion can explain a counter cyclical equity premium, while the conditional volatility of consumption is constant. I extend this reasoning and show that stochastic uncertainty aversion (Knight (1921)) is an alternative channel for the bond market. In the model, the equilibrium bond premium, volatility, and option implied volatility are counter cyclical, while conditional volatility of consumption and inflation is constant. The reason are time-varying uncertainty premiums. Both premiums exist in equilibrium, because the agent faces model uncertainty about the underlying long-run risk dynamics.

The analysis concludes that the yield curve does not span the uncertainty premiums. At the same, these premiums are of first-order importance for understanding bond returns and option markets in this model. The model provides a general equilibrium story for the empirical findings that unspanned factors drive bond premiums, bond volatility, and option implied volatility (Collin-Dufresne and Goldstein (2002), Cochrane and Piazzesi (2005), Ludvigson and Ng (2009)).

I conclude that the introduced small deviation from an otherwise conditionally homoscedastic consumption based asset pricing model has important and realistic implications for bond and bond option markets. For future research it is promising to study stochastic risk aversion and stochastic uncertainty aversion in a unified model.

Appendices

A. Data

Macro data:

The Survey of Professional Forecasters (SPF) does not publish forecasts on consumption growth. I use forecasts and dispersion on GDP growth instead. Real GDP growth, GDP implicit price deflator, federal funds rate are from the St. Louis Fed database (FRED). The quarterly forecast on GDP growth and inflation coincide with the corresponding median forecast from the SPF. For each quarter, I determine the amount of ambiguity, η^2, as the cross-sectional variance of SPF's inflation and GDP growth forecasts. To remove seasonality I use a 4-quarter moving average. All data is from first quarter 1972 to second quarter 2009.

Bond data:

Nominal yields: continuously compounded U.S. government bond yields of maturities 1,2,3,4,5,6,7,8,9,10 years. Data is from first quarter 1972 to second quarter 2009.
Real yields: continuously compounded yields from U.S. Treasury Inflation Protected Securities (TIPS) with maturities of 5,6,7,8,9,10 years. Data is from first quarter 2003 to second quarter 2009.
All bond data is from the Board of Governors of the Federal Reserve System.

Volatility data:

Realized volatility of changes in nominal yields: I use daily squared differences in continuously compounded U.S. government bond yields of maturities 1,2,3,4,5,6,7,8,9,10 years to construct a quarterly measure of realized volatility. Data is from first quarter 1972 to second quarter 2009.

B. Proof of Proposition 1

Rewrite the constrained minimization in (12) as a relative entropy constrained HJB. J denotes the value function. It depends on $J = J(\ln c, \eta^w, \eta^r, z_t)$. The time varying Lagrange multipliers for the entropy constraints are θ_t^r and

θ_t^w.

$$\rho J(\ln c_t, \eta^w, \eta^r, z_t) = \min_{h_t^r, h_t^w} \ln c_t + \theta_t^r \left(\frac{(h_t^r)^2}{2} - A^r (\eta_t^r)^2 \right) + \theta_t^w \left(\frac{(h_t^w)^2}{2} - A^w (\eta_t^w)^2 \right) + \\ + \mathcal{A}^h J(\ln c_t, \eta^w, \eta^r, z_t), \tag{31}$$

where $\mathcal{A}^h$ is the second order differential operator (under the ambiguity adjusted measure) applied to the value function J. Guess the value function is linear in the states, i.e. $J = \delta_0 + \delta_z z_t + \delta_{\eta^w} \eta_t^w + \delta_{\eta^r} \eta_t^r$. The second order differential operator applied to the value function is

$$\mathcal{A}^h\, J = \delta_z(\kappa_z + \sigma_{1z} h_t^r + \sigma_{2z} h_t^w) + \delta_{\eta^r}(a_{\eta^r} + \kappa_{\eta^r} \eta_t^r) + \delta_{\eta^w}(a_{\eta^w} + \kappa_{\eta^w} \eta_t^w) \tag{32}$$

First-order conditions with regard to h_t^r and θ_t^r reveal

$$\theta_t^r = \frac{-\sigma_{1z}\delta_z}{\pm\sqrt{2A^r}\eta_t^r} \tag{33}$$

$$h_t^r = \pm\sqrt{2A^r}\eta_t^r. \tag{34}$$

Note $(\delta_z > 0, \sigma_{1z} > 0)$, the robust HJB is minimized at

$$h_t^r = -\sqrt{2A^r}\eta_t^r \equiv m^r \eta_t^r, \quad m^r \in \mathcal{R}^- \tag{35}$$

$$\theta_t^r = \frac{-\sigma_{1z}\delta_z}{-\sqrt{2A^r}\eta_t^r} \equiv \frac{b_0}{\eta_t^r}, \quad b_0 \in \mathcal{R}^+, \tag{36}$$

where we defined $m^r \equiv -\sqrt{2A_r} < 0$ and $b_0 \equiv \frac{-\sigma_{1z}\delta_z}{-\sqrt{2A^r}} > 0$. This proofs the first part of the proposition.
First-order conditions with regard to h_t^w and θ_t^w reveal

$$\theta_t^w = \frac{-\sigma_{2z}\delta_z}{\pm\sqrt{2A^w}\eta_t^w} \tag{37}$$

$$h_t^w = \pm\sqrt{2A^w}\eta_t^w. \tag{38}$$

Note $(\delta_z > 0, \sigma_{2z} < 0)$, the robust HJB is minimized at

$$h_t^w = \sqrt{2A^w}\eta_t^w \equiv m_w \eta_t^w, \quad m_w \in \mathcal{R}^+ \tag{39}$$

$$\theta_t^w = -\frac{\sigma_{2z}\delta_z}{\sqrt{2A^w}\eta_t^w} \equiv \frac{b_1}{\eta_t^w}, \quad b_1 \in \mathcal{R}^+, \tag{40}$$

where we defined $m_w \equiv \sqrt{2A^w} < 0$ and $b_1 \equiv \frac{-\sigma_{2z}\delta_z}{\sqrt{2A^w}} > 0$. This proofs the remaining part of the proposition. Plug the solution to the robust HJB and verify that the guess of the linearized value function was correct.

C. Derivation of Real Bond Yields

Let $F^r = F_t^r(\tau)$ be the price of a real bond. In the economy this price is exponentially affine $F^r = e^{A^r(\tau)+B^r(\tau)S}$ where S denotes the state vector $S_t = (z_t\, h_t^r\, h_t^w)'$. F^r solves the following PDE

$$r \cdot F^r = \mathcal{A}^H F^r + F_t^r, \quad F_t^r = -F_\tau^r \tag{41}$$

where r is the real risk free rate, $\mathcal{A}^H$ is the second order differential operator and F_τ^r is the first derivative of F^r with regard to τ. Using the equilibrium real interest rate and the exogenous dynamics of S reveals that the bond loadings solve simple ordinary differential equations. The solution for $B_z^r(\tau)$ is $B_z^r(\tau) = \frac{1}{\kappa^z}(1 - e^{\kappa_z \tau})$. The corresponding ode for $B_{h^w}^r$ and $B_{h^w}^r$ solve

$$\frac{d}{d\tau} B_{h^w}^r(\tau) = \kappa_{\eta^w} B_{h^w}^r(\tau) + \frac{1}{2}\sigma_{\eta^w}^2 m^w \left(B_{h^w}^r(\tau)\right)^2 + \sigma_{2z} B_z^r(\tau), B_{h^w}^r(0) = 0 \tag{42}$$

$$\frac{d}{d\tau} B_{h^r}^r(\tau) = \kappa_{\eta^r} B_{h^r}^r(\tau) + \frac{1}{2}\sigma_{\eta^r}^2 m^r \left(B_{h^r}^r(\tau)\right)^2 + \sigma_{1z} B_z^r(\tau), B_{h^r}^r(0) = 0 \tag{43}$$

$$\tag{44}$$

The analytic solution to the Riccati equations (approximating $B_z^r(\tau)$ at its steady state value $B_z^r(\infty)$) is

$$B_{h^r}^r(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - g e^{d\tau})} \tag{45}$$

$$g := \frac{-\beta_1 + d}{-\beta_1 - d} \tag{46}$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \tag{47}$$

$$\beta_0 := \frac{\sigma_{1z}}{\kappa_z} \tag{48}$$

$$\beta_1 := \kappa_{\eta^r} \tag{49}$$

$$\beta_2 := 0.5 m^r \sigma_{\eta^r}^2 \tag{50}$$

and

$$B^r_{h^w}(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - ge^{d\tau})} \tag{51}$$

$$g := \frac{-\beta_1 + d}{-\beta_1 - d} \tag{52}$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \tag{53}$$

$$\beta_0 := \frac{\sigma_{2z}}{\kappa_z} \tag{54}$$

$$\beta_1 := \kappa_{\eta^w} \tag{55}$$

$$\beta_2 := 0.5 m^w \sigma^2_{\eta^w}. \tag{56}$$

Function $A^r(\tau)$ follows from direct integration.

D. Derivation of Nominal Bond Yields

Let $F = F_t(\tau)$ be the price of a nominal bond. In the economy this price is exponentially affine $F = e^{A^n(\tau)+B^n(\tau)X}$ where X denotes the state vector $X_t = (w_t\, S_t)'$. F solves the following PDE

$$R \cdot F = \mathcal{A}^H F + F_t, \quad F_t = -F_\tau \tag{57}$$

where R is the nominal short rate, $\mathcal{A}^H$ is the second order differential operator and F_τ is the first derivative of F with regard to τ. Using the equilibrium nominal short rate and the exogenous dynamics of X reveals that the bond loadings solve simple ordinary differential equations. The solution to the loadings is as follows: $B^n_z(\tau) = B^r_z(\tau)$, $B^n_{h^r}(\tau) = B^r_{h^r}(\tau)$, $B^n_w(\tau) = \frac{1}{\kappa_w}(1 - e^{\kappa_w \tau})$. The analytical solution to $B^n_{h^w}(\tau)$ (approximated $B^n_w(\tau)$ at its steady state value $B^n_w(\tau)$) is

$$B^n_{h^w}(\tau) = \frac{(-\beta_1 + d)(1 - e^{d\tau})}{2\beta_2(1 - ge^{d\tau})} \tag{58}$$

$$g := \frac{-\beta_1 + d}{-\beta_1 - d} \tag{59}$$

$$d := \sqrt{\beta_1^2 - 4\beta_0\beta_2} \tag{60}$$

$$\beta_0 := \frac{\sigma_{2z}}{\kappa_z} + \frac{\sigma_w}{\kappa_w} \tag{61}$$

$$\beta_1 := \kappa_{\eta^w} \tag{62}$$

$$\beta_2 := 0.5 m^w \sigma^2_{\eta^w}. \tag{63}$$

Function $A^n(\tau)$ follows from direct integration.

E. Derivation of Detection Error Probability

The derivation of the detection-error probabilities $p_T(m^r, m^w)$ follows directly from Maenhout (2006):

$$p_T(m^r, m^w) = \frac{1}{2}\left(Pr\left(\ln\frac{dQ_T^h}{dQ_T^0} > 0|dQ^0, \mathcal{F}_0\right) + Pr\left(\ln\frac{dQ_T^0}{dQ_T^h} > 0|dQ^h, \mathcal{F}_0\right)\right) \tag{64}$$

$$\begin{aligned} &= \frac{1}{2}\left(Pr\left(-\frac{1}{2}\int_0^T h_m' h_m dm + \int_0^T h_m \cdot dW_m^z > 0|dQ^0, \mathcal{F}_0\right)\right) \\ &+ \frac{1}{2}\left(Pr\left(-\frac{1}{2}\int_0^T h_m' h_m dm - \int_0^T h_m \cdot dW_m^{z,h} > 0|dQ^h, \mathcal{F}_0\right)\right) \end{aligned} \tag{65}$$

where $h_t = (m^w\eta_t^w \; m^r\eta_t^r)'$ is the endogenous distortion to trend GDP growth. The last equation coincides with

$$p_T(m^r, m^w) = \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \left(Re\left(\frac{\phi^h(k,0,T)}{ik}\right) - Re\left(\frac{\phi(k,0,T)}{ik}\right)\right)dk \tag{66}$$

where $i = \sqrt{-1}$, $\phi(.)$ is defined as $\phi(k,0,T) := E\left[e^{i\cdot k\cdot \xi_{1,T}}|\mathcal{F}_0\right]$ and $\phi^h(.)$ is defined as $\phi^h(k,0,T) := E^h\left[e^{i\cdot k\cdot \xi_{1,T}}|\mathcal{F}_0\right]$ and $\xi_{1,T} = \ln\frac{dQ_T^h}{dQ_T^0}$.

Applying Feynman-Kac theorem to ϕ^h and ϕ reveals that they are an exponentially quadratic function in the amount of inflation distortion h_t:

$$\phi^h(k,t,T) = z_t^{ik+1} e^{G(\tau,k)+\sum_{j\in\{w,r\}} E_j(\tau,k)h_j(t)+\sum_{j\in\{w,r\}}\frac{F_j(\tau,k)}{2}h_j^2(t)} \tag{67}$$

$$\phi(k,t,T) = z_t^{ik} e^{\hat{G}(\tau,k)+\sum_{j\in\{w,r\}} \hat{E}_j(\tau,k)h_j(t)+\sum_{j\in\{w,r\}}\frac{\hat{F}_j(\tau,k)}{2}h_j^2(t)} \tag{68}$$

$$z_T := e^{\xi_{1,T}}, \tag{69}$$

where $G(\tau,k), E_j(\tau,k), F_j(\tau,k), \hat{G}(\tau,k), \hat{E}_j(\tau,k), \hat{F}_j(\tau,k)$ are deterministic solutions to standard complex valued Riccati equations. I provide details on the derivation of the Riccati equations for ϕ^h. The derivation of ϕ is analogous.

In order to get an analytical solution, I approximate the conditional volatility of the uncertainty premium by its steady state value, i.e.

$$dh_t = (a_\eta m + \kappa_\eta h_t)dt + \sigma_\eta \sqrt{m}\sqrt{h_t} dW_t^\eta \tag{70}$$

$$\approx (a_\eta m + \kappa_\eta h_t)dt + \sigma_\eta \sqrt{m}\sqrt{m a_\eta/(-\kappa_\eta)} dW_t^\eta \tag{71}$$

For ease of notation I define $b := \sqrt{m}\sqrt{m a_\eta/(-\kappa_\eta)}$, where more specifically b_r refers to the conditional steady state volatility of dh^r and b_w is the analog for dh^w.

$\phi^h(k,t,T)$ solves $\phi^h_\tau = \mathcal{A}\phi^h$ where $\tau = T-t$ and ϕ^h_τ stands for $\frac{\partial \phi^h}{\partial \tau}$.

$$\phi^h_\tau = \phi^h \left(\dot{G}(\tau,k) + \sum_{j\in\{w,r\}} \dot{E}_j(\tau,k) h_j(t) + \frac{1}{2}\sum_{j\in\{w,r\}} \dot{F}_j(\tau,k) h_j^2(t) \right) \tag{72}$$

$$\frac{\mathcal{A}\phi^h}{\phi^h} = \sum_{j\in\{w,r\}} \left[(E_j(\tau,k) + F_j(\tau,k)h_j(t)) \left(a_{\eta^j} m^j + \kappa_{\eta^j} h_j(t) \right) \right] + 0.5ik(k+1)\left(h_w^2(t) + h_r^2(t) \right)$$

$$+ \frac{1}{2}\sum_{j\in\{w,r\}} \left(E_j^2(\tau,k) + F_j^2(\tau,k) h_j^2(t) + 2E_j(\tau,k)F_j(\tau,k)h_j(t) \right) b_j^2 \tag{73}$$

Set $\phi^h_\tau = \mathcal{A}\phi^h$ and match coefficients:

$$F_j(\tau,k) = F_j^r(\tau,k) + F_j^c(\tau,k) \tag{74}$$

$$F_j^c(\tau,k) = k\cdot\tau \tag{75}$$

$$F_j^r(\tau,k) = \frac{(a_j + d_j)(1 - e^{d_j\tau})}{2b_{2j}^r(1 - g_j e^{d_j \tau})} \tag{76}$$

where F^r is the real part of F and F^c is the complex part and

$$a_j = -b_{1j}^r; \quad d_j = \sqrt{a_j^2 - 4b_{0j}^r b_{2j}^r}; \quad g_j = \frac{a_j + d_j}{a_j - d_j}; \quad b_{0j}^r = -k^2 \tag{77}$$

$$b_{1j}^r = 2\kappa_{\eta^j}; \quad b_{2j}^r = b_j^2 \tag{78}$$

where $j \in \{w,r\}$. The stable steady state solution of F is

$$F_j(\infty,k) = -\frac{b_{1j}^r + d_j}{2b_{2j}^r}. \tag{79}$$

The loadings $E_j(\tau,k), j \in \{f,w,r\}$ solve the following ode

$$\frac{d}{d\tau}E_j(\tau,k) = \kappa_{\eta^j}E_j(\tau,k) + m_j a_{\eta^j}F_j(\tau,k) + E_j(\tau,k)F_j(\tau,k)b_j^2. \tag{80}$$

We obtain an analytical approximation by approximating $F_j(\tau,k)$ around its steady state value $F_j(\infty,k)$.

$$E_j(\tau,k) = -\frac{\hat{a}_j}{\hat{b}_j}(1-e^{\hat{b}_j\tau}) \tag{81}$$

$$\hat{a}_j = F_j(\infty,k)m_j a_{\eta^j} \tag{82}$$

$$\hat{b}_j = F_j(\infty,k)b_j^2 + \kappa_{\eta^j}. \tag{83}$$

The loading $G(\tau,k)$ is obtained through straightforward integration

$$G(\tau,k) = \sum_{j\in\{f,w,r\}}\left(m_j a_{\eta^j}\int_0^\tau E_j(u,k)du\right) + \frac{1}{2}\sum_{j\in\{w,r\}} b_j^2\int_0^\tau E_j^2(u,k)du. \tag{84}$$

The required expression $\phi^h(k,0,T)$ is therefore

$$\phi^h(k,0,T) = e^{G(T,k)+\sum_{j\in\{w,r\}}E_j(T,k)h_j(\infty)+\frac{1}{2}\sum_{j\in\{w,r\}}F_j(T,k)h_j^2(\infty)}, \tag{85}$$

where we assumed that $h_j(0)$ started in its steady state $h_j(\infty) = \frac{m_j a_{\eta^j}}{-\kappa_{\eta^j}}$.

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Table 1: PARAMETER ESTIMATES (Standard Errors)

Panel A: State Variables

	Drift, Volatility			
	κ	σ		a
w	-0.049 (0.0004)	0.0023 (0.00002)		0 (fixed)
z	-0.27 (0.002)	0.0078 (0.00039)	-0.0018 (0.00005)	0 (fixed)
η^w	-0.49 (0.007)	0.78 (0.16)		0.0015 (0.00025)
η^z	-0.22 (0.002)	0.50 (0.95)		0.0017(0.00075)

Panel B: Growth and Inflation

c_0	0.0065 (fixed)
p_0	0.0096 (fixed)
σ_c	8.8e-5 (4e-5)
σ_p	5.4e-5 (4e-5)
ρ	0.001 (fixed)
m^w	40.97 (0.496)
m^r	-0.93 (0.042)

Note: The table presents QML parameter estimates and their standard error (in parenthesis). The asymptotic standard errors are determined based on the score of the log likelihood. The QML estimation uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Table 2: Yield Curve, in %, per quarter

Panel A: Nominal Yields

y^n

maturity	R	4	8	12	16	20	24	28	32	36	40
data	1.6	1.5822	1.6391	1.6819	1.7159	1.7453	1.7717	1.795	1.8154	1.8327	1.8484
model	1.5991	1.5577	1.5915	1.6354	1.6797	1.7215	1.7601	1.7952	1.8270	1.8558	1.8817

Panel B: Real Yields

y^r

maturity	20	24	28	32	36	40
data	0.3782	0.4109	0.4400	0.4650	0.4859	0.5014
model	0.5274	0.5233	0.5204	0.5181	0.5164	0.5150

Panel C: Volatility of Nominal Yield Changes

$\sqrt{Var(\Delta y^n)}$

maturity	4	8	12	16	20	24	28	32	36	40
data	0.1221	0.1321	0.1344	0.1326	0.1298	0.1269	0.1245	0.1226	0.1214	0.1206
model	0.4050	0.2060	0.1385	0.1041	0.0834	0.0695	0.0596	0.0521	0.0464	0.0417

Note: Panel A compares model implied nominal bond yields with the data counterpart. R stands for the nominal short rate (federal funds rate), while the other maturities refer to quarters. The yields are in quarterly percentage units. Panel B compares model implied real bond yields with the data counterpart. Maturity is in quarterly units, and interest rates are in quarterly percentage units. Panel C compares the model implied volatility of nominal yield changes with the data counterpart. Volatilities are in quarterly units and in percent. The QML estimation uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Table 3: Variance Decomposition

Panel A: Nominal Yields

y^n

maturity	total	w	z	h^w	h^r
4	1.0000	0.7358	0.2637	0.0004	0.0000
8	1.0000	0.8397	0.1602	0.0001	0.0000
12	1.0000	0.8937	0.1062	0.0001	0.0000
16	1.0000	0.9228	0.0771	0.0001	0.0000
20	1.0000	0.9395	0.0604	0.0000	0.0000
24	1.0000	0.9498	0.0502	0.0000	0.0000
28	1.0000	0.9565	0.0435	0.0000	0.0000
32	1.0000	0.9611	0.0389	0.0000	0.0000
36	1.0000	0.9643	0.0356	0.0000	0.0000
40	1.0000	0.9667	0.0333	0.0000	0.0000

Panel B: 4-Quarter Excess Holding Period Return of Nominal Bonds

BondPremium

maturity	total	w	z	h^w	h^r
4	1.0000	0	0	0.9313	0.0687
8	1.0000	0	0	0.9769	0.0231
12	1.0000	0	0	0.9886	0.0114
16	1.0000	0	0	0.9929	0.0071
20	1.0000	0	0	0.9950	0.0050
24	1.0000	0	0	0.9961	0.0039
28	1.0000	0	0	0.9967	0.0033
32	1.0000	0	0	0.9972	0.0028
36	1.0000	0	0	0.9975	0.0025
40	1.0000	0	0	0.9977	0.0023

Note: This table summarizes several variance decompositions. Panel A depicts variance decompositions for nominal bond yields. Panel B presents variance decompositions for the 4-quarter model implied expected excess return of holding a nominal bond with different maturities. The QML estimation uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Table 4: Variance Decomposition

Panel C: Implied Volatility for One Quarter Yield Option

$BlackImpliedVolatilities$

maturity	total	w	z	h^w	h^r
4	1.0000	0	0	0.9982	0.0018
8	1.0000	0	0	0.9933	0.0067
12	1.0000	0	0	0.9901	0.0099
16	1.0000	0	0	0.9886	0.0114
20	1.0000	0	0	0.9880	0.0120
24	1.0000	0	0	0.9878	0.0122
28	1.0000	0	0	0.9878	0.0122
32	1.0000	0	0	0.9877	0.0123
36	1.0000	0	0	0.9878	0.0122
40	1.0000	0	0	0.9877	0.0123

Note: Panel C depicts variance decompositions for the Black implied volatility for a three month call on several interest rates. The QML estimation uses bond yield, bond volatility and macro data from 1972.I to 2009.II.

Figure 1: **Term Structure of Nominal Interest Rates**

This figure shows the model implied nominal yield curve (annualized and in %) together with the data counter part. The four factor macro model uses only observable factors in $X = (w\, z\, \eta^w\, \eta^r)'$ (trend inflation, trend GDP growth, dispersion in inflation forecasts, dispersion in GDP growth forecasts), i.e. $y_t^n(\tau) = a^n(\tau) + b^n(\tau)X_t$. The data is from 1972.I to 2009.II. The x-axis is in quarterly units, the y-axis is in percent. The GDP and inflation uncertainty premiums are practically unspanned by the yield curve, which makes the yield curve itself be primarily driven by long-run inflation risk (trend inflation) and long-run GDP risk (trend GDP growth).

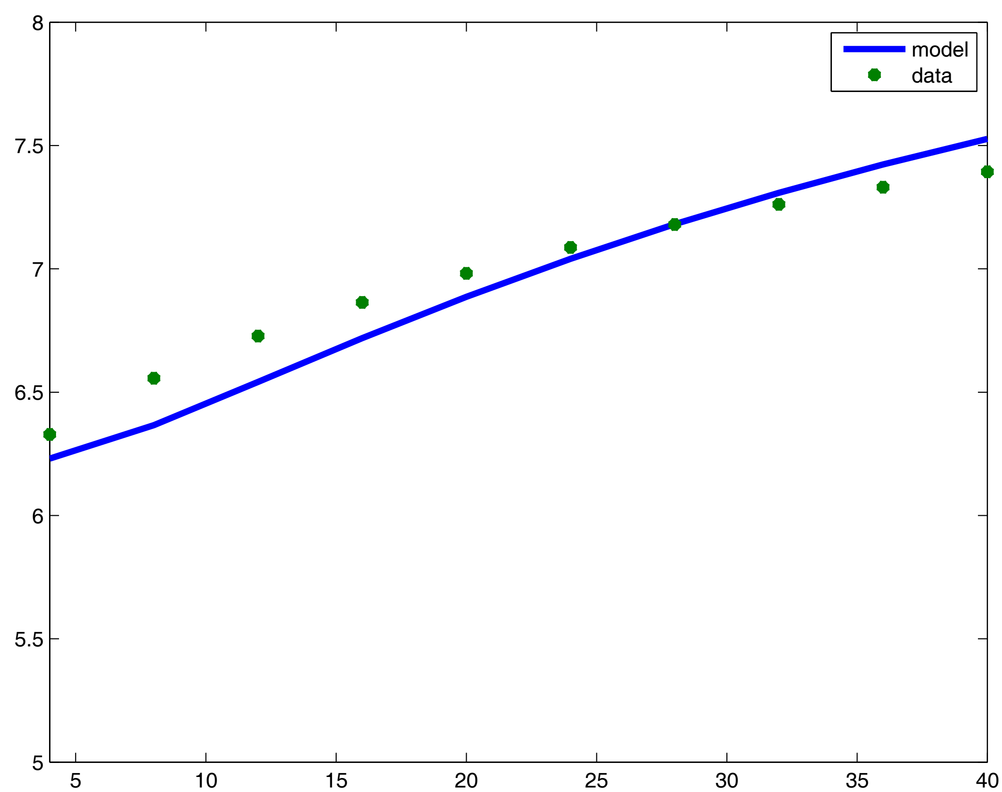

Figure 2: **Impulse Responses: Yield Curve**
This figure presents impulse responses of the yield curve for a one percent increase in the cross-sectional standard deviation of forecasts on long-run risk. The left panels, from top to bottom, analyze the impact of a one percent increase in the cross-sectional standard deviation in inflation forecasts on the nominal short rate, the 40-quarter nominal yield, and the slope of the yield curve. The right panels present the analog for a one percent increase in the cross-sectional dispersion in GDP forecasts. The x-axis are in quarters, the y-axis is in percent. The model is estimated with QML and uses data from 1972.I to 2009.II.

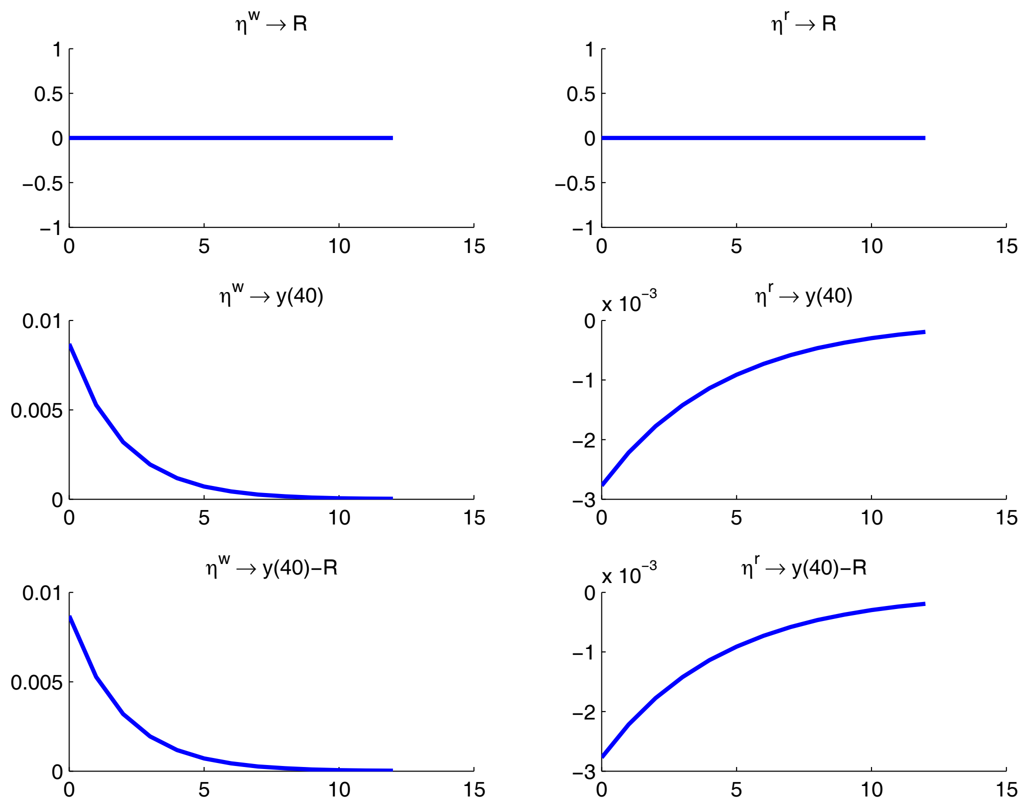

Figure 3: **Term Structure of Bond Premiums**

This figure presents the cross-section of bond premiums in the data and in the model. The bond premium is calculated as the 4-quarter holding period return of all nominal bonds minus the nominal short rate (federal funds rate). The x-axis is in quarterly units and corresponds to the maturity of the nominal bond. The y-axis is in percent. The premiums are annualized. The dotted line corresponds to the realized sample average. The solid line stands for the model implied expected excess bond return. The $-.$ line in the lower panel represents the empirical confidence interval around the realized sample average return. The confidence interval is determined as the realized sample average plus and minus two times the sample standard deviation divided by the square-root of the sample size. It represents the data estimate for the expected bond return in population. The model is estimated with yields and macro data in a one step QML. The data length is 1972.I to 2009.II. The bond premium in the model depends only on the unspanned long-run ambiguity premiums.

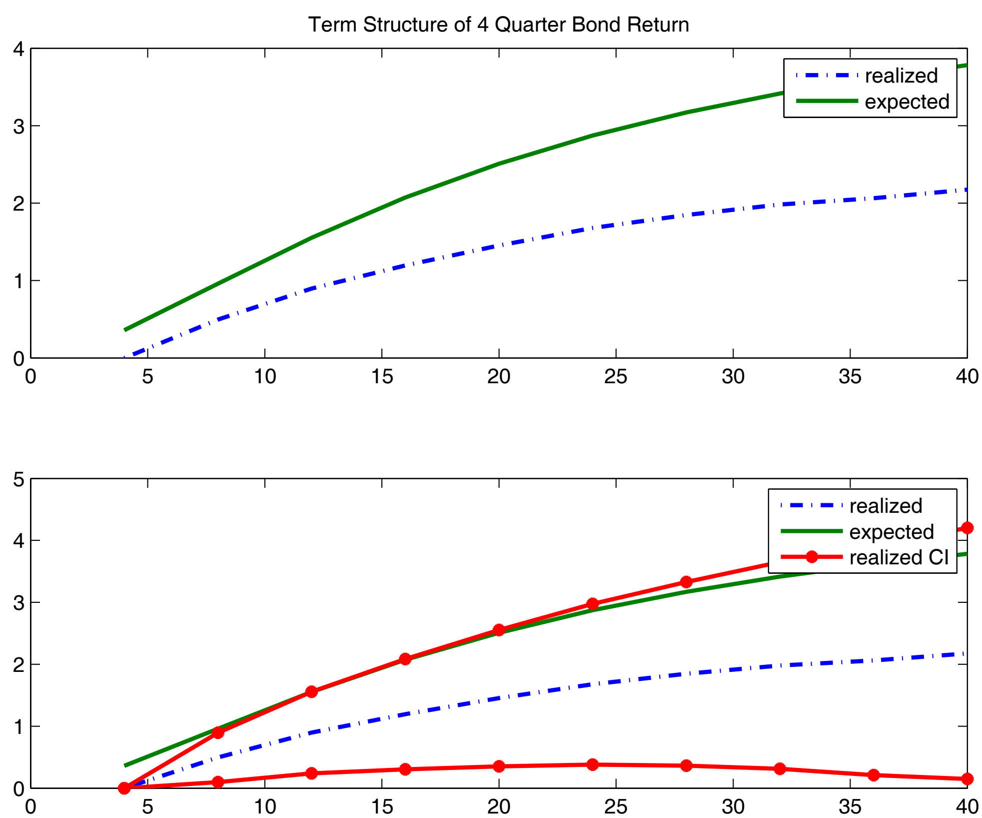

Figure 4: **Time-Series of Bond Premium**

This figure presents the bond premium for the ten year bond and its components together with NBER recession dates. The bond premium is normalized (mean zero and unit variance) and calculated as the 4-quarter holding period return of a ten year bond over the nominal short rate (federal funds rate). The solid line of the top panel presents the realized normalized bond premium, while the dotted line corresponds to the model implied expected normalized bond premium. The lower panel decomposes the model implied expected normalized bond premium for a ten year bond ($*$ line) to its two components. The first component is the unspanned ambiguity factor for trend inflation (solid line, η^w) (cross-sectional standard deviation of inflation forecasts). The second component is the unspanned ambiguity factor for trend GDP growth (dotted line, η^z) (cross-sectional standard deviation of GDP forecasts). The model is estimated with yield and macro data and uses a one step QML estimation with data from 1972.I to 2009.II.

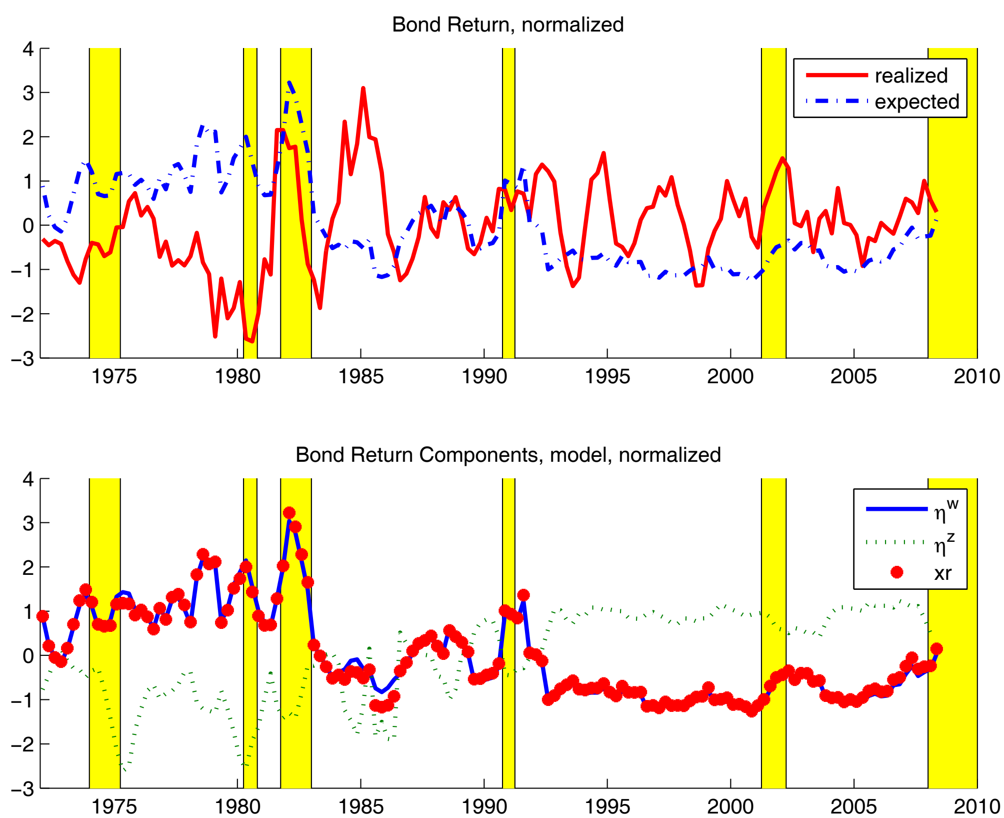

Figure 5: **Impulse Responses: Bond Premium**

This figure presents impulse responses for a one percent increase in the cross-sectional standard deviation of inflation and GDP growth forecasts on itself and on bond premiums. The x-axis is in quarterly units, while the y-axis is in percentage units. The upper panel, from left to right, shows the response of cross-sectional dispersion of inflation forecasts, the bond premium of a 4-quarter bond, and the bond premium of a 40-quarter bond for a one percent increase in the cross-sectional dispersion of inflation forecasts. The lower panel presents the equivalent statistics for a one percent increase in the cross-sectional dispersion of GDP growth forecasts. The model is estimated with yield and macro data and uses a one step QML estimation with data from 1972.I to 2009.II.

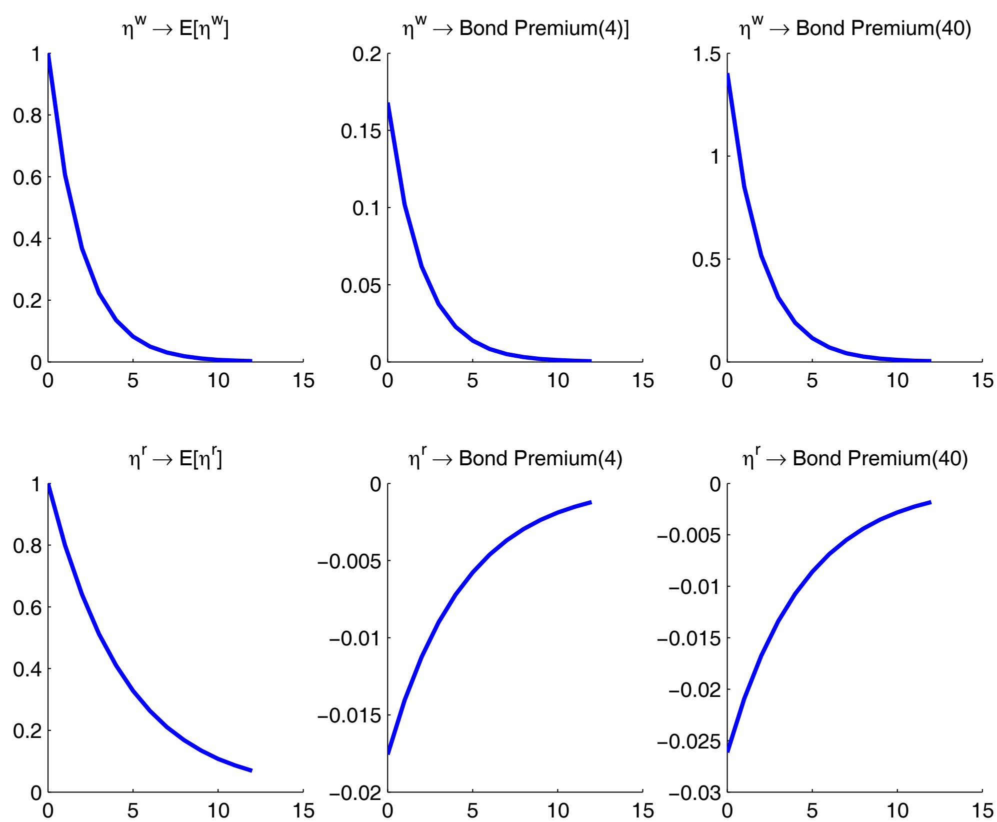

Figure 6: **Term Structure of Yield Volatility**
This figure presents the data and model implied volatility of yield changes. The x-axis stands for the maturity of the particular yield, while the y-axis is in percent. The volatility data is annualized. The solid dotted line represents the sample mean of the corresponding volatility of yield changes. This volatility is determined as the square-root of the average of the sum of squared daily yield changes within a particular quarter. The dotted line represents this sample mean plus and minus two standard deviations. The model is estimated with yield and macro data and uses a one step QML estimation with data from 1972.I to 2009.II.

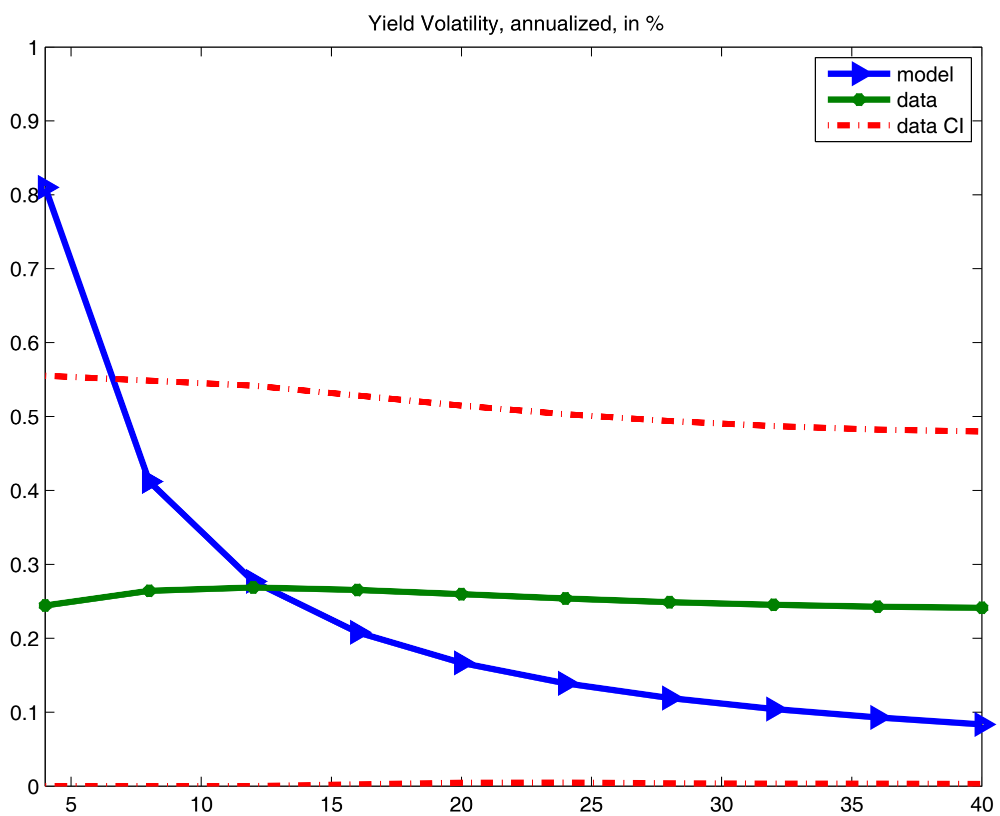

Figure 7: **Impulse Responses: Yield Volatility**

This figure presents the change in the volatility of (nominal) yield changes for a one percent increase in the cross-sectional standard deviation of inflation and GDP growth forecasts. The left panels, from top to bottom, presents the vol response for the 4-quarter and 40-quarter yield to a one percent increase in the cross-sectional standard deviation of inflation forecasts. The right panels present the same responses for a one percent increase in the cross-sectional standard deviation of GDP growth forecasts. The model is estimated with yield and macro data and uses a one step QML estimation with data from 1972.I to 2009.II.

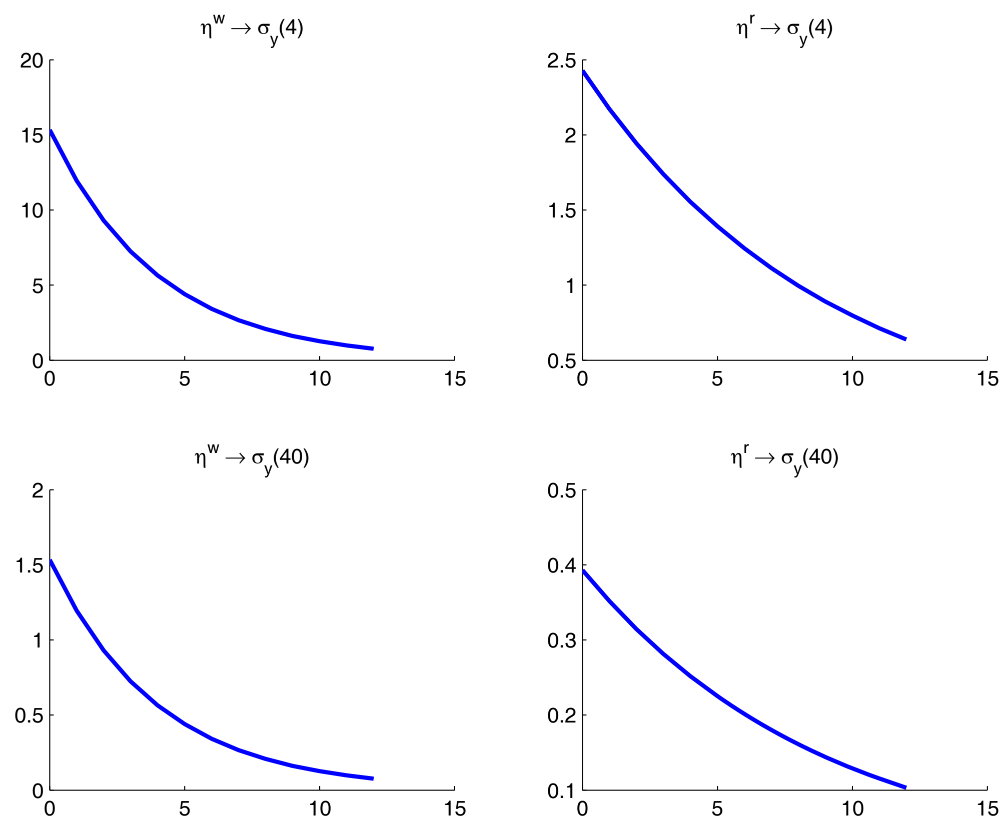

Figure 8: **Option Implied Volatilities**

This figure presents Black implied volatilities a three month call on several Treasury bond yields. The panels from left to right represent the Black implied volatilities for the 8-quarter, 12-quarter, 16-quarter, and 20-quarter yield. The call is priced as $\$100E_t[m^{\$}_{t+1}max(y^n_{t+1}(\tau) - K, 0)]$, where $\tau = (8, 12, 16, 20)$ quarters stands for the maturity of the yield, K is the strike, and $m^{\$}$ is the equilibrium implied nominal stochastic discount factor. The notional of the contract is \$100. The call price is found via monte carlo simulation, with an Euler-Marujama scheme, simulated at a daily interval. In order to get the option implied volatility, I plug the call price into the Black(1973) option formula and invert for the volatility. The skewed smile is entirely driven by the unspanned uncertainty premiums. The x-axis depicts the moneyness, i.e. $\ln \frac{K}{y_t(\tau)}$. The y-axis is in percent. The QML estimates and sample mean of the states are used to construct the pictures. The model is estimated with yield and macro data and uses a one step QML estimation with data from 1972.I to 2009.II.

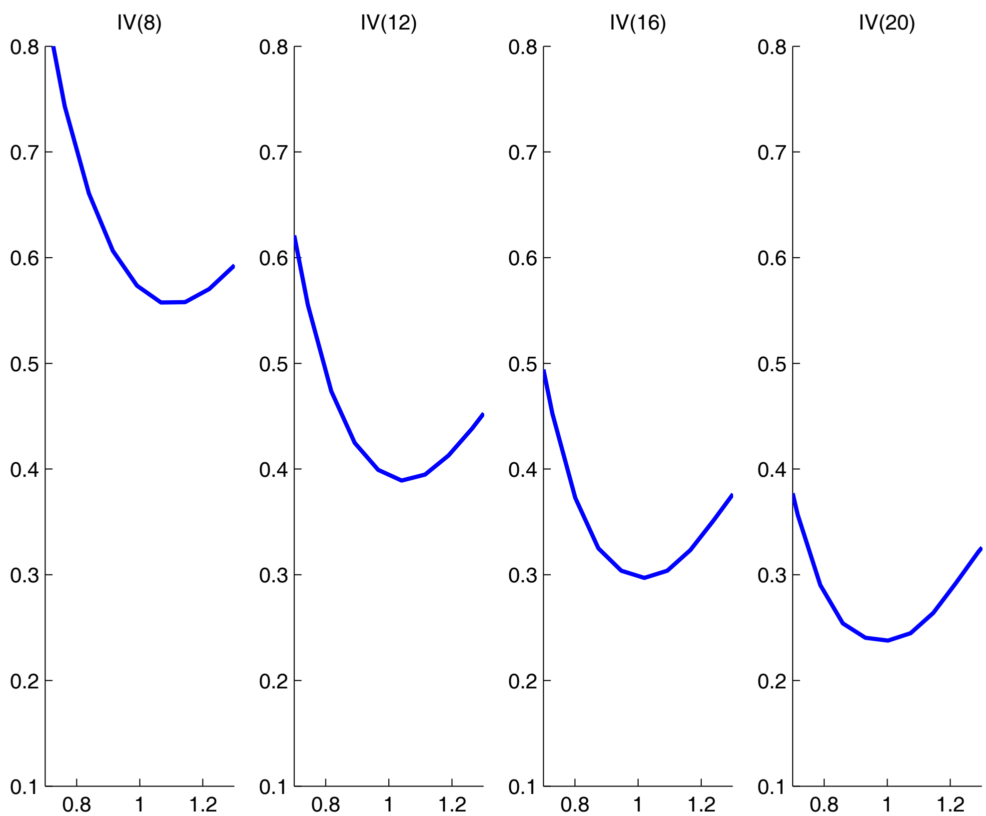

Figure 9: **Impulse Responses: Option Implied Volatilities**
This figure presents responses of option implied volatilities to a one percent increase in the cross-sectional standard deviation of forecasts on next quarter inflation and GDP growth. The upper panels, from left to right, summarize changes in the implied volatility of a three month call on the 8-quarter, 12-quarter, 16-quarter, and 20-quarter yield for a one percent increase in the cross-sectional standard deviation of GDP growth forecasts. The lower panel presents the same statistics for a one percent increase in the cross-sectional standard deviation of inflation forecasts. The call is priced as $\$100E_t[m^{\$}_{t+1}max(y^n_{t+1}(\tau) - K, 0)]$, where $\tau = (8, 12, 16, 20)$ quarters stands for the maturity of the yield, K is the strike, and $m^{\$}$ is the equilibrium implied nominal stochastic discount factor. The notional of the contract is \$100. The corresponding dispersion is increased by one percent, and the call price determined via Monte Carlo simulation, with an Euler-Marujama scheme, simulated at a daily interval. The Black implied volatility is determined. In order to get the implied volatility response to the one percent shock, I subtract the status-quo implied volatility. The x-axis depicts the moneyness, i.e. $\ln \frac{K}{y_t(\tau)}$. The y-axis is in percent. The model is estimated with yield and macro data and uses a one step QML estimation with data from 1972.I to 2009.II.